
C/C++ Library Interfaces

Release 10.5.rc0

The Sage Development Team

Nov 16, 2024

CONTENTS

1	ECL	3
2	eclib	11
3	FLINT	39
4	Giac	45
5	GMP-ECM	49
6	GSL	51
7	lcalc	53
8	libSingular	61
9	GAP	83
10	LinBox	109
11	lrcalc	111
12	mpmath	119
13	NTL	125
14	PARI	127
15	Symmetrica	139
16	Indices and Tables	149
	Python Module Index	151
	Index	153

An underlying philosophy in the development of Sage is that it should provide unified library-level access to the some of the best GPL'd C/C++ libraries. Sage provides access to many libraries which are included with Sage.

The interfaces are implemented via shared libraries and data is moved between systems purely in memory. In particular, there is no interprocess interpreter parsing (e.g., `pexpect`), since everything is linked together and run as a single process. This is much more robust and efficient than using `pexpect`.

Each of these interfaces is used by other parts of Sage. For example, `eclib` is used by the elliptic curves module to compute ranks of elliptic curves and `PARI` is used for computation of class groups. It is thus probably not necessary for a casual user of Sage to be aware of the modules described in this chapter.

1.1 Library interface to Embeddable Common Lisp (ECL)

class `sage.libs.ecl.EclListIterator`

Bases: `object`

Iterator object for an ECL list

This class is used to implement the iterator protocol for `EclObject`. Do not instantiate this class directly but use the `iterator` method on an `EclObject` instead. It is an error if the `EclObject` is not a list.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: I=EclListIterator(EclObject("(1 2 3)"))
sage: type(I)
<class 'sage.libs.ecl.EclListIterator'>
sage: [i for i in I]
[<ECL: 1>, <ECL: 2>, <ECL: 3>]
sage: [i for i in EclObject("(1 2 3)")]
[<ECL: 1>, <ECL: 2>, <ECL: 3>]
sage: EclListIterator(EclObject("1"))
Traceback (most recent call last):
...
TypeError: ECL object is not iterable
```

class `sage.libs.ecl.EclObject`

Bases: `object`

Python wrapper of ECL objects

The `EclObject` forms a wrapper around ECL objects. The wrapper ensures that the data structure pointed to is protected from garbage collection in ECL by installing a pointer to it from a global data structure within the scope of the ECL garbage collector. This pointer is destroyed upon destruction of the `EclObject`.

`EclObject()` takes a Python object and tries to find a representation of it in Lisp.

EXAMPLES:

Python lists get mapped to LISP lists. None and Boolean values to appropriate values in LISP:

```
sage: from sage.libs.ecl import *
sage: EclObject([None,true,false])
<ECL: (NIL T NIL)>
```

Numerical values are translated to the appropriate type in LISP:


```
sage: EclObject([1,2, EclObject([3])])
<ECL: (1 2 (3))>
```

Calling an EclObject translates into the appropriate LISP apply, where the argument is transformed into an EclObject itself, so one can flexibly apply LISP functions:

```
sage: car = EclObject("car")
sage: cdr = EclObject("cdr")
sage: car(cdr([1,2,3]))
<ECL: 2>
```

and even construct and evaluate arbitrary S-expressions:

```
sage: eval=EclObject("eval")
sage: quote=EclObject("quote")
sage: eval([car, [cdr, [quote, [1,2,3]]]])
<ECL: 2>
```

atomp()

Return True if self is atomic, False otherwise.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: EclObject([]).atomp()
True
sage: EclObject([[]]).atomp()
False
```

caar()

Return the caar of self.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>
```

cadr()

Return the cadr of self.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
```

(continues on next page)

(continued from previous page)

```
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>
```

car()

Return the car of self.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>
```

cdar()

Return the cdar of self.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>
```

cddr()

Return the cddr of self.

EXAMPLES:

```

sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>

```

cdr()

Return the cdr of self.

EXAMPLES:

```

sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>

```

characterp()

Return True if self is a character, False otherwise.

Strings are not characters

EXAMPLES:

```

sage: from sage.libs.ecl import *
sage: EclObject('"a"').characterp()
False

```

cons(*d*)

Apply cons to self and argument and return the result.

EXAMPLES:

```

sage: from sage.libs.ecl import *
sage: a=EclObject(1)
sage: b=EclObject(2)
sage: a.cons(b)
<ECL: (1 . 2)>

```

consp()

Return True if self is a cons, False otherwise. NIL is not a cons.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: EclObject([]).consp()
False
sage: EclObject([[[]]).consp()
True
```

eval()

Evaluate object as an S-Expression.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: S=EclObject("(+ 1 2)")
sage: S
<ECL: (+ 1 2)>
sage: S.eval()
<ECL: 3>
```

fixnump()

Return True if self is a fixnum, False otherwise.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: EclObject(2**3).fixnump()
True
sage: EclObject(2**200).fixnump()
False
```

listp()

Return True if self is a list, False otherwise. NIL is a list.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: EclObject([]).listp()
True
sage: EclObject([[[]]).listp()
True
```

nullp()

Return True if self is NIL, False otherwise.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: EclObject([]).nullp()
True
sage: EclObject([[[]]).nullp()
False
```

python()

Convert an EclObject to a python object.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L = EclObject([1,2, ("three", "four")])
sage: L.python()
[1, 2, ('THREE', 'four')]
```

rplaca (*d*)

Destructively replace car(self) with *d*.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L=EclObject((1,2))
sage: L
<ECL: (1 . 2)>
sage: a=EclObject(3)
sage: L.rplaca(a)
sage: L
<ECL: (3 . 2)>
```

rplacd (*d*)

Destructively replace cdr(self) with *d*.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: L=EclObject((1,2))
sage: L
<ECL: (1 . 2)>
sage: a=EclObject(3)
sage: L.rplacd(a)
sage: L
<ECL: (1 . 3)>
```

symbolp ()

Return True if self is a symbol, False otherwise.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: EclObject([]).symbolp()
True
sage: EclObject([[[]]).symbolp()
False
```

`sage.libs.ecl.ecl_eval` (*s*)

Read and evaluate string in Lisp and return the result.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: ecl_eval("(defun fibo (n) (cond((= n 0) 0)((= n 1) 1)(T (+ (fibo (- n 1))
↪ (fibo (- n 2)))))")
<ECL: FIBO>
sage: ecl_eval("(mapcar 'fibo '(1 2 3 4 5 6 7))")
<ECL: (1 1 2 3 5 8 13)>
```

`sage.libs.ecl.init_ecl()`

Internal function to initialize ecl. Do not call.

This function initializes the ECL library for use within Python. This routine should only be called once and importing the ecl library interface already does that, so do not call this yourself.

EXAMPLES:

```
sage: from sage.libs.ecl import *
```

At this point, `init_ecl()` has run. Explicitly executing it gives an error:

```
sage: init_ecl()
Traceback (most recent call last):
...
RuntimeError: ECL is already initialized
```

`sage.libs.ecl.print_objects()`

Print GC-protection list.

Diagnostic function. ECL objects that are bound to Python objects need to be protected from being garbage collected. We do this by including them in a doubly linked list bound to the global ECL symbol *SAGE-LIST-OF-OBJECTS*. Only non-immediate values get included, so small integers do not get linked in. This routine prints the values currently stored.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: a=EclObject("hello")
sage: b=EclObject(10)
sage: c=EclObject("world")
sage: print_objects() #random because previous test runs can have left objects
NIL
WORLD
HELLO
```

`sage.libs.ecl.shutdown_ecl()`

Shut down ecl. Do not call.

Given the way that ECL is used from python, it is very difficult to ensure that no ECL objects exist at a particular time. Hence, destroying ECL is a risky proposition.

EXAMPLES:

```
sage: from sage.libs.ecl import *
sage: shutdown_ecl()
```

`sage.libs.ecl.test_ecl_options()`

Print an overview of the ECL options.

`sage.libs.ecl.test_sigint_before_ecl_sig_on()`

2.1 Sage interface to Cremona's `eclib` library (also known as `mwrnk`)

This is the Sage interface to John Cremona's `eclib` C++ library for arithmetic on elliptic curves. The classes defined in this module give Sage interpreter-level access to some of the functionality of `eclib`. For most purposes, it is not necessary to directly use these classes. Instead, one can create an `EllipticCurve` and call methods that are implemented using this module.

Note

This interface is a direct library-level interface to `eclib`, including the 2-descent program `mwrnk`.

class `sage.libs.eclib.interface.mwrnk_EllipticCurve` (*ainvs*, *verbose=False*)

Bases: `SageObject`

The `mwrnk_EllipticCurve` class represents an elliptic curve using the `Curvedata` class from `eclib`, called here an 'mwrnk elliptic curve'.

Create the mwrnk elliptic curve with invariants `ainvs`, which is a list of 5 or less *integers* a_1, a_2, a_3, a_4 , and a_5 .

If strictly less than 5 invariants are given, then the *first* ones are set to 0, so, e.g., `[3, 4]` means $a_1 = a_2 = a_3 = 0$ and $a_4 = 3, a_5 = 4$.

INPUT:

- `ainvs` – list or tuple list of 5 or less integers, the coefficients of a nonsingular Weierstrass equation
- `verbose` – boolean (default: `False`); verbosity flag. If `True`, then all Selmer group computations will be verbose.

EXAMPLES:

We create the elliptic curve $y^2 + y = x^3 + x^2 - 2x$:

```
sage: e = mwrnk_EllipticCurve([0, 1, 1, -2, 0])
sage: e.ainvs()
[0, 1, 1, -2, 0]
```

This example illustrates that omitted *a*-invariants default to 0:

```
sage: e = mwrnk_EllipticCurve([3, -4])
sage: e
```

(continues on next page)

(continued from previous page)

```

y^2 = x^3 + 3 x - 4
sage: e.ainvs()
[0, 0, 0, 3, -4]

```

The entries of the input list are coerced to `int`. If this is impossible, then an error is raised:

```

sage: e = mwrank_EllipticCurve([3, -4.8]); e
Traceback (most recent call last):
...
TypeError: ainvs must be a list or tuple of integers.

```

When you enter a singular model you get an exception:

```

sage: e = mwrank_EllipticCurve([0, 0])
Traceback (most recent call last):
...
ArithmeticError: Invariants (= 0,0,0,0,0) do not describe an elliptic curve.

```

`CPS_height_bound()`

Return the Cremona-Prickett-Siksek height bound. This is a floating point number B such that if P is a point on the curve, then the naive logarithmic height $h(P)$ is less than $B + \hat{h}(P)$, where $\hat{h}(P)$ is the canonical height of P .

Warning

We assume the model is minimal!

EXAMPLES:

```

sage: E = mwrank_EllipticCurve([0, 0, 0, -1002231243161, 0])
sage: E.CPS_height_bound()
14.163198527061496
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: E.CPS_height_bound()
0.0

```

`ainvs()`

Return the a -invariants of this mwrank elliptic curve.

EXAMPLES:

```

sage: E = mwrank_EllipticCurve([0,0,1,-1,0])
sage: E.ainvs()
[0, 0, 1, -1, 0]

```

`certain()`

Return `True` if the last `two_descent()` call provably correctly computed the rank. If `two_descent()` hasn't been called, then it is first called by `certain()` using the default parameters.

The result is `True` if and only if the results of the methods `rank()` and `rank_bound()` are equal.

EXAMPLES:

A 2-descent does not determine $E(\mathbf{Q})$ with certainty for the curve $y^2 + y = x^3 - x^2 - 120x - 2183$:


```
sage: E = mwrank_EllipticCurve([0, -1, 1, -120, -2183])
sage: E.two_descent(False)
...
sage: E.certain()
False
sage: E.rank()
0
```

The previous value is only a lower bound; the upper bound is greater:

```
sage: E.rank_bound()
2
```

In fact the rank of E is actually 0 (as one could see by computing the L -function), but Sha has order 4 and the 2-torsion is trivial, so mwrank cannot conclusively determine the rank in this case.

conductor()

Return the conductor of this curve, computed using Cremona's implementation of Tate's algorithm.

Note

This is independent of PARI's.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([1, 1, 0, -6958, -224588])
sage: E.conductor()
2310
```

gens()

Return a list of the generators for the Mordell-Weil group.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0, 0, 1, -1, 0])
sage: E.gens()
[[0, -1, 1]]
```

isogeny_class(verbose=False)

Return the isogeny class of this mwrank elliptic curve.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0, -1, 1, 0, 0])
sage: E.isogeny_class()
([[0, -1, 1, 0, 0], [0, -1, 1, -10, -20], [0, -1, 1, -7820, -263580]], [[0, 5,
↪ 0], [5, 0, 5], [0, 5, 0]])
```

rank()

Return the rank of this curve, computed using `two_descent()`.

In general this may only be a lower bound for the rank; an upper bound may be obtained using the function `rank_bound()`. To test whether the value has been proved to be correct, use the method `certain()`.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.rank()
0
sage: E.certain()
True
```

```
sage: E = mwrank_EllipticCurve([0, -1, 1, -929, -10595])
sage: E.rank()
0
sage: E.certain()
False
```

rank_bound()

Return an upper bound for the rank of this curve, computed using *two_descent()*.

If the curve has no 2-torsion, this is equal to the 2-Selmer rank. If the curve has 2-torsion, the upper bound may be smaller than the bound obtained from the 2-Selmer rank minus the 2-rank of the torsion, since more information is gained from the 2-isogenous curve or curves.

EXAMPLES:

The following is the curve 960D1, which has rank 0, but Sha of order 4:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.rank_bound()
0
sage: E.rank()
0
```

In this case the rank was computed using a second descent, which is able to determine (by considering a 2-isogenous curve) that Sha is nontrivial. If we deliberately stop the second descent, the rank bound is larger:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.two_descent(second_descent = False, verbose=False)
sage: E.rank_bound()
2
```

In contrast, for the curve 571A, also with rank 0 and Sha of order 4, we only obtain an upper bound of 2:

```
sage: E = mwrank_EllipticCurve([0, -1, 1, -929, -10595])
sage: E.rank_bound()
2
```

In this case the value returned by *rank()* is only a lower bound in general (though this is correct):

```
sage: E.rank()
0
sage: E.certain()
False
```

regulator()

Return the regulator of the saturated Mordell-Weil group.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0, 0, 1, -1, 0])
sage: E.regulator()
0.051111140823996884
```

saturate (*bound=-1, lower=2*)

Compute the saturation of the Mordell-Weil group.

INPUT:

- *bound* – integer (default: -1); if -1, saturate at *all* primes by computing a bound on the saturation index, otherwise saturate at all primes up to the minimum of *bound* and the saturation index bound
- *lower* – integer (default: 2); only saturate at primes not less than this

EXAMPLES:

Since the 2-descent automatically saturates at primes up to 20, further saturation often has no effect:

```
sage: E = mwrank_EllipticCurve([0, 0, 0, -1002231243161, 0])
sage: E.gens()
[[-1001107, -4004428, 1]]
sage: E.saturate()
sage: E.gens()
[[-1001107, -4004428, 1]]
```

Check that [Issue #18031](#) is fixed:

```
sage: E = EllipticCurve([0, -1, 1, -266, 968])
sage: Q1 = E([-1995, 3674, 125])
sage: Q2 = E([157, 1950, 1])
sage: E.saturation([Q1, Q2])
([(1 : -27 : 1), (157 : 1950 : 1)], 3, 0.801588644684981)
```

selmer_rank ()

Return the rank of the 2-Selmer group of the curve.

EXAMPLES:

The following is the curve 960D1, which has rank 0, but Sha of order 4. The 2-torsion has rank 2, and the Selmer rank is 3:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.selmer_rank()
3
```

Nevertheless, we can obtain a tight upper bound on the rank since a second descent is performed which establishes the 2-rank of Sha:

```
sage: E.rank_bound()
0
```

To show that this was resolved using a second descent, we do the computation again but turn off `second_descent`:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.two_descent(second_descent = False, verbose=False)
sage: E.rank_bound()
2
```

For the curve 571A, also with rank 0 and Sha of order 4, but with no 2-torsion, the Selmer rank is strictly greater than the rank:

```

sage: E = mwrank_EllipticCurve([0, -1, 1, -929, -10595])
sage: E.selmer_rank()
2
sage: E.rank_bound()
2

```

In cases like this with no 2-torsion, the rank upper bound is always equal to the 2-Selmer rank. If we ask for the rank, all we get is a lower bound:

```

sage: E.rank()
0
sage: E.certain()
False

```

`set_verbose(verbose)`

Set the verbosity of printing of output by the `two_descent()` and other functions.

INPUT:

- `verbose` – integer; if positive, print lots of output when doing 2-descent

EXAMPLES:

```

sage: E = mwrank_EllipticCurve([0, 0, 1, -1, 0])
sage: E.saturate() # no output
sage: E.gens()
[[0, -1, 1]]

sage: E = mwrank_EllipticCurve([0, 0, 1, -1, 0])
sage: E.set_verbose(1)
sage: E.saturate() # tol 1e-10
Basic pair: I=48, J=-432
disc=255744
2-adic index bound = 2
By Lemma 5.1(a), 2-adic index = 1
2-adic index = 1
One (I,J) pair
Looking for quartics with I = 48, J = -432
Looking for Type 2 quartics:
Trying positive a from 1 up to 1 (square a first...)
(1,0,-6,4,1) --trivial
Trying positive a from 1 up to 1 (...then non-square a)
Finished looking for Type 2 quartics.
Looking for Type 1 quartics:
Trying positive a from 1 up to 2 (square a first...)
(1,0,0,4,4) --nontrivial...(x:y:z) = (1 : 1 : 0)
Point = [0:0:1]
height = 0.0511114082399688402358
Rank of B=im(eps) increases to 1 (The previous point is on the egg)
Exiting search for Type 1 quartics after finding one which is globally_
↳soluble.
Mordell rank contribution from B=im(eps) = 1
Selmer rank contribution from B=im(eps) = 1
Sha rank contribution from B=im(eps) = 0
Mordell rank contribution from A=ker(eps) = 0
Selmer rank contribution from A=ker(eps) = 0
Sha rank contribution from A=ker(eps) = 0
Searching for points (bound = 8)...done:

```

(continues on next page)

(continued from previous page)

```

found points which generate a subgroup of rank 1
and regulator 0.0511114082399688402358
Processing points found during 2-descent...done:
now regulator = 0.0511114082399688402358
Saturating (with bound = -1)...done:
points were already saturated.

```

silverman_bound()

Return the Silverman height bound. This is a floating point number B such that if P is a point on the curve, then the naive logarithmic height $h(P)$ is less than $B + \hat{h}(P)$, where $\hat{h}(P)$ is the canonical height of P .

Warning

We assume the model is minimal!

EXAMPLES:

```

sage: E = mwrnk_EllipticCurve([0, 0, 0, -1002231243161, 0])
sage: E.silverman_bound()
18.29545210468247
sage: E = mwrnk_EllipticCurve([0, 0, 1, -7, 6])
sage: E.silverman_bound()
6.284833369972403

```

two_descent (*verbose=True, selmer_only=False, first_limit=20, second_limit=8, n_aux=-1, second_descent=True*)

Compute 2-descent data for this curve.

INPUT:

- `verbose` – boolean (default: `True`); print what mwrnk is doing
- `selmer_only` – boolean (default: `False`); selmer_only switch
- `first_limit` – integer (default: 20); bound on $|x| + |z|$ in quartic point search
- `second_limit` – integer (default: 8); bound on $\log \max(|x|, |z|)$, i.e. logarithmic
- `n_aux` – integer (default: -1); (only relevant for general 2-descent when 2-torsion trivial) number of primes used for quartic search. `n_aux=-1` causes default (8) to be used. Increase for curves of higher rank.
- `second_descent` – boolean (default: `True`); (only relevant for curves with 2-torsion, where mwrnk uses descent via 2-isogeny) flag determining whether or not to do second descent. *Default strongly recommended.*

OUTPUT: nothing

class `sage.libs.eclib.interface.mwrnk_MordellWeil` (*curve, verbose=True, pp=1, maxr=999*)

Bases: `SageObject`

The `mwrnk_MordellWeil` class represents a subgroup of a Mordell-Weil group. Use this class to saturate a specified list of points on an `mwrnk_EllipticCurve`, or to search for points up to some bound.

INPUT:

- `curve` – `mwrnk_EllipticCurve`; the underlying elliptic curve
- `verbose` – boolean (default: `False`); verbosity flag (controls amount of output produced in point searches)

- `pp` – integer (default: 1); process points flag (if nonzero, the points found are processed, so that at all times only a **Z**-basis for the subgroup generated by the points found so far is stored. If zero, no processing is done and all points found are stored).
- `maxr` – integer (default: 999); maximum rank (quit point searching once the points found generate a subgroup of this rank. Useful if an upper bound for the rank is already known).

EXAMPLES:

```

sage: E = mwrank_EllipticCurve([1,0,1,4,-6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ
Subgroup of Mordell-Weil group: []
sage: EQ.search(2)
P1 = [0:1:0]      is torsion point, order 1
P1 = [1:-1:1]    is torsion point, order 2
P1 = [2:2:1]     is torsion point, order 3
P1 = [9:23:1]    is torsion point, order 6

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(2)
P1 = [0:1:0]      is torsion point, order 1
P1 = [-3:0:1]     is generator number 1
...
P4 = [-91:804:343] = -2*P1 + 2*P2 + 1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]

```

Example to illustrate the verbose parameter:

```

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E, verbose=False)
sage: EQ.search(1)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]

sage: EQ = mwrank_MordellWeil(E, verbose=True)
sage: EQ.search(1)
P1 = [0:1:0]      is torsion point, order 1
P1 = [-3:0:1]     is generator number 1
saturating up to 20...Saturation index bound (for points of good reduction) = 3
Reducing saturation bound from given value 20 to computed index bound 3
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 7)
Checking 3-saturation
Points were proved 3-saturated (max q used = 7)
done
P2 = [-2:3:1]     is generator number 2
saturating up to 20...Saturation index bound (for points of good reduction) = 4
Reducing saturation bound from given value 20 to computed index bound 4
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
possible kernel vector = [1,1]
This point may be in 2E(Q): [14:-52:1]

```

(continues on next page)

(continued from previous page)

```

...and it is!
Replacing old generator #1 with new generator [1:-1:1]
Reducing index bound from 4 to 2
Points have successfully been 2-saturated (max q used = 7)
Index gain = 2^1
done, index = 2.
Gained index 2, new generators = [ [1:-1:1] [-2:3:1] ]
P3 = [-14:25:8] is generator number 3
saturating up to 20...Saturation index bound (for points of good reduction) = 3
Reducing saturation bound from given value 20 to computed index bound 3
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done, index = 1.
P4 = [-1:3:1] = -1*P1 + -1*P2 + -1*P3 (mod torsion)
P4 = [0:2:1] = 2*P1 + 0*P2 + 1*P3 (mod torsion)
P4 = [2:13:8] = -3*P1 + 1*P2 + -1*P3 (mod torsion)
P4 = [1:0:1] = -1*P1 + 0*P2 + 0*P3 (mod torsion)
P4 = [2:0:1] = -1*P1 + 1*P2 + 0*P3 (mod torsion)
P4 = [18:7:8] = -2*P1 + -1*P2 + -1*P3 (mod torsion)
P4 = [3:3:1] = 1*P1 + 0*P2 + 1*P3 (mod torsion)
P4 = [4:6:1] = 0*P1 + -1*P2 + -1*P3 (mod torsion)
P4 = [36:69:64] = 1*P1 + -2*P2 + 0*P3 (mod torsion)
P4 = [68:-25:64] = -2*P1 + -1*P2 + -2*P3 (mod torsion)
P4 = [12:35:27] = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]

```

Example to illustrate the process points (pp) parameter:

```

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E, verbose=False, pp=1)
sage: EQ.search(1); EQ # generators only
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
sage: EQ = mwrank_MordellWeil(E, verbose=False, pp=0)
sage: EQ.search(1); EQ # all points found
Subgroup of Mordell-Weil group: [[-3:0:1], [-2:3:1], [-14:25:8], [-1:3:1], ↵
↵ [0:2:1], [2:13:8], [1:0:1], [2:0:1], [18:7:8], [3:3:1], [4:6:1], [36:69:64], ↵
↵ [68:-25:64], [12:35:27]]

```

points()

Return a list of the generating points in this Mordell-Weil group.

OUTPUT: list of lists of length 3, each holding the primitive integer coordinates $[x, y, z]$ of a generating point

EXAMPLES:

```

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(1)
P1 = [0:1:0] is torsion point, order 1
P1 = [-3:0:1] is generator number 1
...
P4 = [12:35:27] = 1*P1 + -1*P2 + -1*P3 (mod torsion)

```

(continues on next page)

(continued from previous page)

```
sage: EQ.points()
[[1, -1, 1], [-2, 3, 1], [-14, 25, 8]]
```

process (*v*, *saturation_bound=0*)

Process points in the list *v*.

This function allows one to add points to a *mwrnk_MordellWeil* object.

INPUT:

- *v* – list of 3-tuples or lists of ints or Integers; a list of triples of integers, which define points on the curve
- *saturation_bound* – integer (default: 0); saturate at primes up to *saturation_bound*, or at *all* primes if *saturation_bound* is -1. When *saturation_bound* is 0 (the default), do no saturation.

OUTPUT:

None. But note that if the *verbose* flag is set, then there will be some output as a side-effect.

EXAMPLES:

```
sage: E = mwrnk_EllipticCurve([0,0,1,-7,6])
sage: E.gens()
[[1, -1, 1], [-2, 3, 1], [-14, 25, 8]]
sage: EQ = mwrnk_MordellWeil(E)
sage: EQ.process([[1, -1, 1], [-2, 3, 1], [-14, 25, 8]])
P1 = [1:-1:1]          is generator number 1
P2 = [-2:3:1]         is generator number 2
P3 = [-14:25:8]       is generator number 3
```

```
sage: EQ.points()
[[1, -1, 1], [-2, 3, 1], [-14, 25, 8]]
```

Example to illustrate the saturation parameter *saturation_bound*:

```
sage: E = mwrnk_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrnk_MordellWeil(E)
sage: EQ.process([[1547, -2967, 343], [2707496766203306, 864581029138191,
↪2969715140223272], [-13422227300, -49322830557, 12167000000]], saturation_
↪bound=20)
P1 = [1547:-2967:343]          is generator number 1
...
Gained index 5, new generators = [ [-2:3:1] [-14:25:8] [1:-1:1] ]
sage: EQ.points()
[[-2, 3, 1], [-14, 25, 8], [1, -1, 1]]
```

Here the processing was followed by saturation at primes up to 20. Now we prevent this initial saturation:

```
sage: E = mwrnk_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrnk_MordellWeil(E)
sage: EQ.process([[1547, -2967, 343], [2707496766203306, 864581029138191,
↪2969715140223272], [-13422227300, -49322830557, 12167000000]], saturation_
↪bound=0)
P1 = [1547:-2967:343]          is generator number 1
P2 = [2707496766203306:864581029138191:2969715140223272] is generator
↪number 2
```

(continues on next page)

(continued from previous page)

```

P3 = [-13422227300:-49322830557:12167000000]          is generator number 3
sage: EQ.points()
[[1547, -2967, 343], [2707496766203306, 864581029138191, 2969715140223272], [-
↪-13422227300, -49322830557, 12167000000]]
sage: EQ.regulator()
375.42920288254555
sage: EQ.saturate(2) # points were not 2-saturated
saturating basis...Saturation index bound (for points of good reduction) = 93
Only p-saturating for p up to given value 2.
The resulting points may not be p-saturated for p between this and the
↪computed index bound 93
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 ]
Checking 2-saturation
possible kernel vector = [1,0,0]
This point may be in 2E(Q): [1547:-2967:343]
...and it is!
Replacing old generator #1 with new generator [-2:3:1]
Reducing index bound from 93 to 46
Points have successfully been 2-saturated (max q used = 11)
Index gain = 2^1
done
Gained index 2
New regulator = 93.85730072
(True, 2, '[ ]')
sage: EQ.points()
[[-2, 3, 1], [2707496766203306, 864581029138191, 2969715140223272], [-
↪-13422227300, -49322830557, 12167000000]]
sage: EQ.regulator()
93.85730072063639
sage: EQ.saturate(3) # points were not 3-saturated
saturating basis...Saturation index bound (for points of good reduction) = 46
Only p-saturating for p up to given value 3.
The resulting points may not be p-saturated for p between this and the
↪computed index bound 46
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
possible kernel vector = [0,1,0]
This point may be in 3E(Q):
↪[2707496766203306:864581029138191:2969715140223272]
...and it is!
Replacing old generator #2 with new generator [-14:25:8]
Reducing index bound from 46 to 15
Points have successfully been 3-saturated (max q used = 13)
Index gain = 3^1
done
Gained index 3
New regulator = 10.42858897
(True, 3, '[ ]')
sage: EQ.points()
[[-2, 3, 1], [-14, 25, 8], [-13422227300, -49322830557, 12167000000]]
sage: EQ.regulator()
10.4285889689596
sage: EQ.saturate(5) # points were not 5-saturated

```

(continues on next page)

(continued from previous page)

```

saturating basis...Saturation index bound (for points of good reduction) = 15
Only p-saturating for p up to given value 5.
The resulting points may not be p-saturated for p between this and the
↳computed index bound 15
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 5 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
Checking 5-saturation
possible kernel vector = [0,0,1]
This point may be in 5E(Q): [-13422227300:-49322830557:12167000000]
...and it is!
Replacing old generator #3 with new generator [1:-1:1]
Reducing index bound from 15 to 3
Points have successfully been 5-saturated (max q used = 71)
Index gain = 5^1
done
Gained index 5
New regulator = 0.4171435588
(True, 5, '[ ]')
sage: EQ.points()
[[-2, 3, 1], [-14, 25, 8], [1, -1, 1]]
sage: EQ.regulator()
0.417143558758384
sage: EQ.saturate() # points are now saturated
saturating basis...Saturation index bound (for points of good reduction) = 3
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done
(True, 1, '[ ]')

```

rank()

Return the rank of this subgroup of the Mordell-Weil group.

OUTPUT: integer

EXAMPLES:

```

sage: E = mwrank_EllipticCurve([0,-1,1,0,0])
sage: E.rank()
0

```

A rank 3 example:

```

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.rank()
0
sage: EQ.regulator()
1.0

```

The preceding output is correct, since we have not yet tried to find any points on the curve either by searching

or 2-descent:

```
sage: EQ
Subgroup of Mordell-Weil group: []
```

Now we do a very small search:

```
sage: EQ.search(1)
P1 = [0:1:0]      is torsion point, order 1
P1 = [-3:0:1]    is generator number 1
saturating up to 20...Checking 2-saturation
...
P4 = [12:35:27]  = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
sage: EQ.rank()
3
sage: EQ.regulator()
0.417143558758384
```

We do in fact now have a full Mordell-Weil basis.

regulator()

Return the regulator of the points in this subgroup of the Mordell-Weil group.

Note

`eclib` can compute the regulator to arbitrary precision, but the interface currently returns the output as a float.

OUTPUT:

(float) The regulator of the points in this subgroup.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0,-1,1,0,0])
sage: E.regulator()
1.0

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: E.regulator()
0.417143558758384
```

saturate (*max_prime=-1, min_prime=2*)

Saturate this subgroup of the Mordell-Weil group.

INPUT:

- `max_prime` – integer (default: -1); if -1, an upper bound is computed for the primes at which the subgroup may not be saturated, and saturation is performed for all primes up to this bound. Otherwise, the bound used is the minimum of `max_prime` and the computed bound.
- `min_prime` – integer (default: 2); only do saturation at primes no less than this. (For example, if the points have been found via `two_descent()` they should already be 2-saturated so a value of 3 is appropriate.)

OUTPUT:

(3-tuple) (ok, index, unsatlist) where:

- `ok` – boolean; `True` if and only if the saturation was provably successful at all primes attempted. If the default was used for `max_prime`, then `True` indicates that the subgroup is saturated at *all* primes.
- `index` – integer; the index of the group generated by the original points in their saturation
- `unsatlist` – list of ints list of primes at which saturation could not be proved or achieved

Note

In versions up to v20190909, `eclib` used floating point methods based on elliptic logarithms to divide points, and did not compute the precision necessary, which could cause failures. Since v20210310, `eclib` uses exact method based on division polynomials, which should mean that such failures does not happen.

Note

We emphasize that if this function returns `True` as the first return argument (`ok`), and if the default was used for the parameter `max_prime`, then the points in the basis after calling this function are saturated at *all* primes, i.e., saturating at the primes up to `max_prime` are sufficient to saturate at all primes. Note that the function computes an upper bound for the index of saturation, and does no work for primes greater than this even if `max_prime` is larger.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0, 0, 1, -7, 6])
sage: EQ = mwrank_MordellWeil(E)
```

We initialise with three points which happen to be 2, 3 and 5 times the generators of this rank 3 curve. To prevent automatic saturation at this stage we set the parameter `sat` to 0 (which is in fact the default):

```
sage: EQ.process([[1547, -2967, 343], [2707496766203306, 864581029138191, ↵
↵2969715140223272], [-13422227300, -49322830557, 12167000000]], saturation_
↵bound=0)
P1 = [1547:-2967:343] is generator number 1
P2 = [2707496766203306:864581029138191:2969715140223272] is generator_
↵number 2
P3 = [-13422227300:-49322830557:12167000000] is generator number 3
sage: EQ
Subgroup of Mordell-Weil group: [[1547:-2967:343], ↵
↵[2707496766203306:864581029138191:2969715140223272], [-13422227300:-
↵49322830557:12167000000]]
sage: EQ.regulator()
375.42920288254555
```

Now we saturate at $p = 2$, and gain index 2:

```
sage: EQ.saturate(2) # points were not 2-saturated
saturating basis...Saturation index bound (for points of good reduction) = 93
Only p-saturating for p up to given value 2.
...
Gained index 2
New regulator = 93.857...
(True, 2, '[ ]')
```

(continues on next page)

(continued from previous page)

```

sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1],
↪ [2707496766203306:864581029138191:2969715140223272], [-13422227300:-
↪ 49322830557:12167000000]]
sage: EQ.regulator()
93.85730072063639

```

Now we saturate at $p = 3$, and gain index 3:

```

sage: EQ.saturate(3) # points were not 3-saturated
saturating basis...Saturation index bound (for points of good reduction) = 46
Only p-saturating for p up to given value 3.
...
Gained index 3
New regulator = 10.428...
(True, 3, '[ ]')
sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1], [-14:25:8], [-13422227300:-
↪ 49322830557:12167000000]]
sage: EQ.regulator()
10.4285889689596

```

Now we saturate at $p = 5$, and gain index 5:

```

sage: EQ.saturate(5) # points were not 5-saturated
saturating basis...Saturation index bound (for points of good reduction) = 15
Only p-saturating for p up to given value 5.
...
Gained index 5
New regulator = 0.417...
(True, 5, '[ ]')
sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1], [-14:25:8], [1:-1:1]]
sage: EQ.regulator()
0.417143558758384

```

Finally we finish the saturation. The output here shows that the points are now provably saturated at all primes:

```

sage: EQ.saturate() # points are now saturated
saturating basis...Saturation index bound (for points of good reduction) = 3
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done
(True, 1, '[ ]')

```

Of course, the `process()` function would have done all this automatically for us:

```

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.process([[1547, -2967, 343], [2707496766203306, 864581029138191,
↪ 2969715140223272], [-13422227300, -49322830557, 12167000000]], saturation_
↪ bound=5)
P1 = [1547:-2967:343] is generator number 1

```

(continues on next page)

(continued from previous page)

```

...
Gained index 5, new generators = [ [-2:3:1] [-14:25:8] [1:-1:1] ]
sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1], [-14:25:8], [1:-1:1]]
sage: EQ.regulator()
0.417143558758384

```

But we would still need to use the `saturate()` function to verify that full saturation has been done:

```

sage: EQ.saturate()
saturating basis...Saturation index bound (for points of good reduction) = 3
Tamagawa index primes are [ 2 ]...
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done
(True, 1, '[ ]')

```

Note the output of the preceding command: it proves that the index of the points in their saturation is at most 3, then proves saturation at 2 and at 3, by reducing the points modulo all primes of good reduction up to 11, respectively 13.

search (*height_limit=18, verbose=False*)

Search for new points, and add them to this subgroup of the Mordell-Weil group.

INPUT:

- `height_limit`—float (default: 18); search up to this logarithmic height

Note

On 32-bit machines, this *must* be < 21.48 ($31 \log(2)$) else $\exp(h_{\text{lim}}) > 2^{31}$ and overflows. On 64-bit machines, it must be *at most* 43.668 ($63 \log(2)$). However, this bound is a logarithmic bound and increasing it by just 1 increases the running time by (roughly) $\exp(1.5) = 4.5$, so searching up to even 20 takes a very long time.

Note

The search is carried out with a quadratic sieve, using code adapted from a version of Michael Stoll's `ratpoints` program. It would be preferable to use a newer version of `ratpoints`.

- `verbose` – boolean (default: `False`) turn verbose operation on or off

EXAMPLES:

A rank 3 example, where a very small search is sufficient to find a Mordell-Weil basis:

```

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(1)
P1 = [0:1:0] is torsion point, order 1

```

(continues on next page)

(continued from previous page)

```
P1 = [-3:0:1]          is generator number 1
...
P4 = [12:35:27]       = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
```

In the next example, a search bound of 12 is needed to find a non-torsion point:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -18392, -1186248]) #1056g4
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(11); EQ
P1 = [0:1:0]          is torsion point, order 1
P1 = [161:0:1]        is torsion point, order 2
Subgroup of Mordell-Weil group: []
sage: EQ.search(12); EQ
P1 = [0:1:0]          is torsion point, order 1
P1 = [161:0:1]        is torsion point, order 2
P1 = [4413270:10381877:27000] is generator number 1
...
Subgroup of Mordell-Weil group: [[4413270:10381877:27000]]
```

2.2 Cython interface to Cremona's `eclib` library (also known as `mwrank`)

EXAMPLES:

```
sage: from sage.libs.eclib.mwrank import _Curvedata, _mw
sage: c = _Curvedata(1,2,3,4,5)

sage: print(c)
[1,2,3,4,5]
b2 = 9      b4 = 11      b6 = 29      b8 = 35
c4 = -183    c6 = -3429
disc = -10351 (# real components = 1)
#torsion not yet computed

sage: t = _mw(c)
sage: t.search(10)
sage: t
[[1:2:1]]
```

`sage.libs.eclib.mwrank.get_precision()`

Return the working floating point bit precision of `mwrank`, which is equal to the global NTL real number precision.

OUTPUT: integer; the current precision in bits

See also `set_precision()`.

EXAMPLES:

```
sage: mwrank_get_precision()
150
```

`sage.libs.eclib.mwrank.initprimes` (*filename*, *verb=False*)

Initialises mwrank/eclib's internal prime list.

INPUT:

- *filename* – string; the name of a file of primes
- *verb* – boolean (default: False); verbose or not

EXAMPLES:

```
sage: import tempfile
sage: with tempfile.NamedTemporaryFile(mode='w+t') as f:
.....:     data = ' '.join(str(p) for p in prime_range(10^7, 10^7 + 20))
.....:     _ = f.write(data)
.....:     f.flush()
.....:     mwrank_initprimes(f.name, verb=True)
Computed 78519 primes, largest is 1000253
reading primes from file ...
read extra prime 10000019
finished reading primes from file ...
Extra primes in list: 10000019

sage: mwrank_initprimes(f.name, True)
Traceback (most recent call last):
...
OSError: No such file or directory: ...
```

`sage.libs.eclib.mwrank.parse_point_list` (*s*)

Parse a string representing a list of points.

INPUT:

- *s* – string representation of a list of points; for example `[]`, `[[1:2:3]]`, or `[[1:2:3],[4:5:6]]`

OUTPUT: list of triples of integers, for example `[]`, `[[1,2,3]]`, `[[1,2,3],[4,5,6]]`

EXAMPLES:

```
sage: from sage.libs.eclib.mwrank import parse_point_list
sage: parse_point_list('[]')
[]
sage: parse_point_list('[[1:2:3]]')
[[1, 2, 3]]
sage: parse_point_list('[[1:2:3],[4:5:6]]')
[[1, 2, 3], [4, 5, 6]]
```

`sage.libs.eclib.mwrank.set_precision` (*n*)

Set the working floating point bit precision of mwrank, which is equal to the global NTL real number precision.

NTL real number bit precision. This has a massive effect on the speed of mwrank calculations. The default (used if this function is not called) is $n=150$, but it might have to be increased if a computation fails.

INPUT:

- *n* – positive integer; the number of bits of precision

Warning

This change is global and affects *all* future calls of eclib functions by Sage.

Note

The minimal value to which the precision may be set is 53. Lower values will be increased to 53.

See also `get_precision()`.

EXAMPLES:

```
sage: from sage.libs.eclib.mwrank import set_precision, get_precision
sage: old_prec = get_precision(); old_prec
150
sage: set_precision(50)
sage: get_precision()
53
sage: set_precision(old_prec)
sage: get_precision()
150
```

2.3 Cremona matrices

class `sage.libs.eclib.mat.Matrix`

Bases: object

A Cremona Matrix.

EXAMPLES:

```
sage: M = CremonaModularSymbols(225)
sage: t = M.hecke_matrix(2)
sage: type(t)
<class 'sage.libs.eclib.mat.Matrix'>
sage: t
61 x 61 Cremona matrix over Rational Field
```

add_scalar (*s*)

Return new matrix obtained by adding *s* to each diagonal entry of `self`.

EXAMPLES:

```
sage: M = CremonaModularSymbols(23, cuspidal=True, sign=1)
sage: t = M.hecke_matrix(2); print(t.str())
[ 0  1]
[ 1 -1]
sage: w = t.add_scalar(3); print(w.str())
[3  1]
[1  2]
```

charpoly (*var='x'*)

Return the characteristic polynomial of this matrix, viewed as as a matrix over the integers.

ALGORITHM:

Note that currently, this function converts this matrix into a dense matrix over the integers, then calls the charpoly algorithm on that, which I think is LinBox's.

EXAMPLES:

```

sage: M = CremonaModularSymbols(33, cuspidal=True, sign=1)
sage: t = M.hecke_matrix(2)
sage: t.charpoly()
x^3 + 3*x^2 - 4
sage: t.charpoly().factor()
(x - 1) * (x + 2)^2

```

ncols()

Return the number of columns of this matrix.

EXAMPLES:

```

sage: M = CremonaModularSymbols(1234, sign=1)
sage: t = M.hecke_matrix(3); t.ncols()
156
sage: M.dimension()
156

```

nrows()

Return the number of rows of this matrix.

EXAMPLES:

```

sage: M = CremonaModularSymbols(19, sign=1)
sage: t = M.hecke_matrix(13); t
2 x 2 Cremona matrix over Rational Field
sage: t.nrows()
2

```

sage_matrix_over_ZZ (*sparse=True*)

Return corresponding Sage matrix over the integers.

INPUT:

- `sparse` – boolean (default: `True`); whether the return matrix has a sparse representation

EXAMPLES:

```

sage: M = CremonaModularSymbols(23, cuspidal=True, sign=1)
sage: t = M.hecke_matrix(2)
sage: s = t.sage_matrix_over_ZZ(); s
[ 0  1]
[ 1 -1]
sage: type(s)
<class 'sage.matrix.matrix_integer_sparse.Matrix_integer_sparse'>
sage: s = t.sage_matrix_over_ZZ(sparse=False); s
[ 0  1]
[ 1 -1]
sage: type(s)
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>

```

str()

Return full string representation of this matrix, never in compact form.

EXAMPLES:

```

sage: M = CremonaModularSymbols(22, sign=1)
sage: t = M.hecke_matrix(13)

```

(continues on next page)

(continued from previous page)

```
sage: t.str()
'[14 0 0 0 0]\n[-4 12 0 8 4]\n[ 0 -6 4 -6 0]\n[ 4 2 0 6 -4]\n[ 0 -
↪0 0 0 14]'
```

class sage.libs.eclib.mat.**MatrixFactory**

Bases: object

2.4 Modular symbols using eclib newforms

class sage.libs.eclib.newforms.**ECModularSymbol**

Bases: object

Modular symbol associated with an elliptic curve, using John Cremona's newforms class.

EXAMPLES:

```
sage: from sage.libs.eclib.newforms import ECModularSymbol
sage: E = EllipticCurve('11a')
sage: M = ECModularSymbol(E,1); M
Modular symbol with sign 1 over Rational Field attached to Elliptic Curve defined_
↪by y^2 + y = x^3 - x^2 - 10*x - 20 over Rational Field
```

By default, symbols are based at the cusp ∞ , i.e. we evaluate $\{\infty, r\}$:

```
sage: [M(1/i) for i in range(1,11)]
[2/5, -8/5, -3/5, 7/5, 12/5, 12/5, 7/5, -3/5, -8/5, 2/5]
```

We can also switch the base point to the cusp 0:

```
sage: [M(1/i, base_at_infinity=False) for i in range(1,11)]
[0, -2, -1, 1, 2, 2, 1, -1, -2, 0]
```

For the minus symbols this makes no difference since $\{0, \infty\}$ is in the plus space. Note that to evaluate minus symbols the space must be defined with sign 0, which makes both signs available:

```
sage: M = ECModularSymbol(E,0); M
Modular symbol with sign 0 over Rational Field attached to Elliptic Curve defined_
↪by y^2 + y = x^3 - x^2 - 10*x - 20 over Rational Field
sage: [M(1/i, -1) for i in range(1,11)]
[0, 0, 1, 1, 0, 0, -1, -1, 0, 0]
sage: [M(1/i, -1, base_at_infinity=False) for i in range(1,11)]
[0, 0, 1, 1, 0, 0, -1, -1, 0, 0]
```

If the ECModularSymbol is created with sign 0 then as well as asking for both + and - symbols, we can also obtain both (as a tuple). However it is more work to create the full modular symbol space:

```
sage: E = EllipticCurve('11a1')
sage: M = ECModularSymbol(E,0); M
Modular symbol with sign 0 over Rational Field attached to Elliptic Curve defined_
↪by y^2 + y = x^3 - x^2 - 10*x - 20 over Rational Field
sage: [M(1/i) for i in range(2,11)]
[[-8/5, 0],
 [-3/5, 1],
 [7/5, 1],
```

(continues on next page)

(continued from previous page)

```
[12/5, 0],
[12/5, 0],
[7/5, -1],
[-3/5, -1],
[-8/5, 0],
[2/5, 0]]
```

The curve is automatically converted to its minimal model:

```
sage: E = EllipticCurve([0,0,0,0,1/4])
sage: EModularSymbol(E)
Modular symbol with sign 1 over Rational Field attached to Elliptic Curve defined_
↪by y^2 + y = x^3 over Rational Field
```

Non-optimal curves are handled correctly in eclib, by comparing the ratios of real and/or imaginary periods:

```
sage: from sage.libs.eclib.newforms import EModularSymbol
sage: E1 = EllipticCurve('11a1') # optimal
sage: E1.period_lattice().basis()
(1.26920930427955, 0.634604652139777 + 1.45881661693850*I)
sage: M1 = EModularSymbol(E1,0)
sage: M1(0)
[2/5, 0]
sage: M1(1/3)
[-3/5, 1]
```

One non-optimal curve has real period 1/5 that of the optimal one, so plus symbols scale up by a factor of 5 while minus symbols are unchanged:

```
sage: E2 = EllipticCurve('11a2') # not optimal
sage: E2.period_lattice().basis()
(0.253841860855911, 0.126920930427955 + 1.45881661693850*I)
sage: M2 = EModularSymbol(E2,0)
sage: M2(0)
[2, 0]
sage: M2(1/3)
[-3, 1]
sage: all((M2(r,1)==5*M1(r,1)) for r in QQ.range_by_height(10))
True
sage: all((M2(r,-1)==M1(r,-1)) for r in QQ.range_by_height(10))
True
```

The other non-optimal curve has real period 5 times that of the optimal one, so plus symbols scale down by a factor of 5; again, minus symbols are unchanged:

```
sage: E3 = EllipticCurve('11a3') # not optimal
sage: E3.period_lattice().basis()
(6.34604652139777, 3.17302326069888 + 1.45881661693850*I)
sage: M3 = EModularSymbol(E3,0)
sage: M3(0)
[2/25, 0]
sage: M3(1/3)
[-3/25, 1]
sage: all((5*M3(r,1)==M1(r,1)) for r in QQ.range_by_height(10))
True
sage: all((M3(r,-1)==M1(r,-1)) for r in QQ.range_by_height(10))
True
```

2.5 Cremona modular symbols

class `sage.libs.eclib.homspace.ModularSymbols`

Bases: `object`

Class of Cremona Modular Symbols of given level and sign (and weight 2).

EXAMPLES:

```
sage: M = CremonaModularSymbols(225)
sage: type(M)
<class 'sage.libs.eclib.homspace.ModularSymbols'>
```

dimension ()

Return the dimension of this modular symbols space.

EXAMPLES:

```
sage: M = CremonaModularSymbols(1234, sign=1)
sage: M.dimension()
156
```

hecke_matrix (*p*, *dual=False*, *verbose=False*)

Return the matrix of the *p*-th Hecke operator acting on this space of modular symbols.

The result of this command is not cached.

INPUT:

- *p* – a prime number
- *dual* – boolean (default: `False`); whether to compute the Hecke operator acting on the dual space, i.e., the transpose of the Hecke operator
- *verbose* – boolean (default: `False`); print verbose output

OUTPUT:

(matrix) If *p* divides the level, the matrix of the Atkin-Lehner involution W_p at *p*; otherwise the matrix of the Hecke operator T_p ,

EXAMPLES:

```
sage: M = CremonaModularSymbols(37)
sage: t = M.hecke_matrix(2); t
5 x 5 Cremona matrix over Rational Field
sage: print(t.str())
[ 3  0  0  0  0]
[-1 -1  1  1  0]
[ 0  0 -1  0  1]
[-1  1  0 -1 -1]
[ 0  0  1  0 -1]
sage: t.charpoly().factor()
(x - 3) * x^2 * (x + 2)^2
sage: print(M.hecke_matrix(2, dual=True).str())
[ 3 -1  0 -1  0]
[ 0 -1  0  1  0]
[ 0  1 -1  0  1]
[ 0  1  0 -1  0]
[ 0  0  1 -1 -1]
```

(continues on next page)

(continued from previous page)

```

sage: w = M.hecke_matrix(37); w
5 x 5 Cremona matrix over Rational Field
sage: w.charpoly().factor()
(x - 1)^2 * (x + 1)^3
sage: sw = w.sage_matrix_over_ZZ()
sage: st = t.sage_matrix_over_ZZ()
sage: sw^2 == sw.parent()(1)
True
sage: st*sw == sw*st
True

```

is_cuspidal()

Return whether or not this space is cuspidal.

EXAMPLES:

```

sage: M = CremonaModularSymbols(1122); M.is_cuspidal()
0
sage: M = CremonaModularSymbols(1122, cuspidal=True); M.is_cuspidal()
1

```

level()

Return the level of this modular symbols space.

EXAMPLES:

```

sage: M = CremonaModularSymbols(1234, sign=1)
sage: M.level()
1234

```

number_of_cusps()

Return the number of cusps for $\Gamma_0(N)$, where N is the level.

EXAMPLES:

```

sage: M = CremonaModularSymbols(225)
sage: M.number_of_cusps()
24

```

sign()

Return the sign of this Cremona modular symbols space. The sign is either 0, +1 or -1.

EXAMPLES:

```

sage: M = CremonaModularSymbols(1122, sign=1); M
Cremona Modular Symbols space of dimension 224 for Gamma_0(1122) of weight 2
↳with sign 1
sage: M.sign()
1
sage: M = CremonaModularSymbols(1122); M
Cremona Modular Symbols space of dimension 433 for Gamma_0(1122) of weight 2
↳with sign 0
sage: M.sign()
0
sage: M = CremonaModularSymbols(1122, sign=-1); M
Cremona Modular Symbols space of dimension 209 for Gamma_0(1122) of weight 2
↳with sign -1

```

(continues on next page)

(continued from previous page)

```
sage: M.sign()
-1
```

sparse_hecke_matrix (*p*, *dual=False*, *verbose=False*, *base_ring='ZZ'*)

Return the matrix of the p -th Hecke operator acting on this space of modular symbols as a sparse Sage matrix over *base_ring*. This is more memory-efficient than creating a Cremona matrix and then applying `sage_matrix_over_ZZ` with `sparse=True`.

The result of this command is not cached.

INPUT:

- *p* – a prime number
- *dual* – boolean (default: `False`); whether to compute the Hecke operator acting on the dual space, i.e., the transpose of the Hecke operator
- *verbose* – boolean (default: `False`); print verbose output

OUTPUT:

(matrix) If p divides the level, the matrix of the Atkin-Lehner involution W_p at p ; otherwise the matrix of the Hecke operator T_p ,

EXAMPLES:

```
sage: M = CremonaModularSymbols(37)
sage: t = M.sparse_hecke_matrix(2); type(t)
<class 'sage.matrix.matrix_integer_sparse.Matrix_integer_sparse'>
sage: print(t)
[ 3  0  0  0  0]
[-1 -1  1  1  0]
[ 0  0 -1  0  1]
[-1  1  0 -1 -1]
[ 0  0  1  0 -1]
sage: M = CremonaModularSymbols(5001)
sage: T = M.sparse_hecke_matrix(2)
sage: U = M.hecke_matrix(2).sage_matrix_over_ZZ(sparse=True)
sage: print(T == U)
True
sage: T = M.sparse_hecke_matrix(2, dual=True)
sage: print(T == U.transpose())
True
sage: T = M.sparse_hecke_matrix(2, base_ring=GF(7))
sage: print(T == U.change_ring(GF(7)))
True
```

This concerns an issue reported on [Issue #21303](#):

```
sage: C = CremonaModularSymbols(45, cuspidal=True, sign=-1)
sage: T2a = C.hecke_matrix(2).sage_matrix_over_ZZ()
sage: T2b = C.sparse_hecke_matrix(2)
sage: print(T2a == T2b)
True
```

2.6 Cremona modular symbols

`sage.libs.eclib.constructor.CremonaModularSymbols` (*level*, *sign=0*, *cuspidal=False*, *verbose=0*)

Return the space of Cremona modular symbols with given level, sign, etc.

INPUT:

- *level* – integer; at least 2, not just positive!
- *sign* – integer (default: 0); either 0 or 1 or -1
- *cuspidal* – boolean (default: False); if True, compute only the cuspidal subspace
- *verbose* – boolean (default: False); if True, print verbose information while creating space

EXAMPLES:

```
sage: M = CremonaModularSymbols(43); M
Cremona Modular Symbols space of dimension 7 for Gamma_0(43) of weight 2 with
↳sign 0
sage: M = CremonaModularSymbols(43, sign=1); M
Cremona Modular Symbols space of dimension 4 for Gamma_0(43) of weight 2 with
↳sign 1
sage: M = CremonaModularSymbols(43, cuspidal=True); M
Cremona Cuspidal Modular Symbols space of dimension 6 for Gamma_0(43) of weight 2
↳with sign 0
sage: M = CremonaModularSymbols(43, cuspidal=True, sign=1); M
Cremona Cuspidal Modular Symbols space of dimension 3 for Gamma_0(43) of weight 2
↳with sign 1
```

When run interactively, the following command will display verbose output:

```
sage: M = CremonaModularSymbols(43, verbose=1)
After 2-term relations, ngens = 22
ngens      = 22
maxnumrel = 32
relation matrix has = 704 entries...
Finished 3-term relations: numrel = 16 ( maxnumrel = 32)
relmat has 42 nonzero entries (density = 0.0596591)
Computing kernel...
time to compute kernel = (... seconds)
rk = 7
Number of cusps is 2
ncusps = 2
About to compute matrix of delta
delta matrix done: size 2x7.
About to compute kernel of delta
done
Finished constructing homspace.
sage: M
Cremona Modular Symbols space of dimension 7 for Gamma_0(43) of weight 2 with
↳sign 0
```

The input must be valid or a `ValueError` is raised:

```
sage: M = CremonaModularSymbols(-1)
Traceback (most recent call last):
...
ValueError: the level (= -1) must be at least 2
```

(continues on next page)

(continued from previous page)

```
sage: M = CremonaModularSymbols(0)
Traceback (most recent call last):
...
ValueError: the level (= 0) must be at least 2
```

The sign can only be 0 or 1 or -1:

```
sage: M = CremonaModularSymbols(10, sign = -2)
Traceback (most recent call last):
...
ValueError: sign (= -2) is not supported; use 0, +1 or -1
```

We do allow -1 as a sign (see [Issue #9476](#)):

```
sage: CremonaModularSymbols(10, sign = -1)
Cremona Modular Symbols space of dimension 0 for Gamma_0(10) of weight 2 with
↳ sign -1
```


3.1 FLINT fmpz_poly class wrapper

AUTHORS:

- Robert Bradshaw (2007-09-15) Initial version.
- William Stein (2007-10-02) update for new flint; add arithmetic and creation of coefficients of arbitrary size.

class `sage.libs.flint.fmpz_poly_sage.Fmpz_poly`

Bases: `SageObject`

Construct a new `fmpz_poly` from a sequence, constant coefficient, or string (in the same format as it prints).

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: Fmpz_poly([1,2,3])
3 1 2 3
sage: Fmpz_poly(5)
1 5
sage: Fmpz_poly(str(Fmpz_poly([3,5,7])))
3 3 5 7
```

degree()

The degree of `self`.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,2,3]); f
3 1 2 3
sage: f.degree()
2
sage: Fmpz_poly(range(1000)).degree()
999
sage: Fmpz_poly([2,0]).degree()
0
```

derivative()

Return the derivative of `self`.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,2,6])
sage: f.derivative().list() == [2, 12]
True
```

div_rem (*other*)

Return $\text{self} / \text{other}$, $\text{self} \% \text{other}$.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,3,4,5])
sage: g = f^23
sage: g.div_rem(f)[1]
0
sage: g.div_rem(f)[0] - f^22
0
sage: f = Fmpz_poly([1..10])
sage: g = Fmpz_poly([1,3,5])
sage: q, r = f.div_rem(g)
sage: q*f+r
17 1 2 3 4 4 4 10 11 17 18 22 26 30 23 26 18 20
sage: g
3 1 3 5
sage: q*g+r
10 1 2 3 4 5 6 7 8 9 10
```

left_shift (*n*)

Left shift self by n .

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,2])
sage: f.left_shift(1).list() == [0,1,2]
True
```

list ()

Return self as a list of coefficients, lowest terms first.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([2,1,0,-1])
sage: f.list()
[2, 1, 0, -1]
```

pow_truncate (*exp*, *n*)

Return self raised to the power of exp mod x^n .

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,2])
sage: f.pow_truncate(10,3)
3 1 20 180
sage: f.pow_truncate(1000,3)
3 1 2000 1998000
```

pseudo_div (*other*)

pseudo_div_rem (*other*)

right_shift (*n*)

Right shift `self` by *n*.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,2])
sage: f.right_shift(1).list() == [2]
True
```

truncate (*n*)

Return the truncation of `self` at degree *n*.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly_sage import Fmpz_poly
sage: f = Fmpz_poly([1,1])
sage: g = f**10; g
11 1 10 45 120 210 252 210 120 45 10 1
sage: g.truncate(5)
5 1 10 45 120 210
```

3.2 File: `sage/libs/flint/fmpz_poly_sage.pyx` (starting at line 1)

3.3 FLINT Arithmetic Functions

`sage.libs.flint.arith_sage.bell_number` (*n*)

Return the *n*-th Bell number.

See [Wikipedia article Bell_number](#).

ALGORITHM:

Uses `arith_bell_number()`.

EXAMPLES:

```
sage: from sage.libs.flint.arith_sage import bell_number
sage: [bell_number(i) for i in range(10)]
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
sage: bell_number(10)
115975
sage: bell_number(40)
157450588391204931289324344702531067
sage: bell_number(100)
475853912767648336587907688413872078263636696868256114666163346375591144978924426226727240442177
```

`sage.libs.flint.arith_sage.bernoulli_number` (*n*)

Return the *n*-th Bernoulli number.

See [Wikipedia article Bernoulli_number](#).

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import bernoulli_number
sage: [bernoulli_number(i) for i in range(10)]
[1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0]
sage: bernoulli_number(10)
5/66
sage: bernoulli_number(40)
-261082718496449122051/13530
sage: bernoulli_number(100)
-
↪94598037819122125295227433069493721872702841533066936133385696204311395415197247711/
↪33330

```

`sage.libs.flint.arith_sage.dedekind_sum(p, q)`

Return the Dedekind sum $s(p, q)$ where p and q are arbitrary integers.

See [Wikipedia article Dedekind_sum](#).

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import dedekind_sum
sage: dedekind_sum(4, 5)
-1/5

```

`sage.libs.flint.arith_sage.euler_number(n)`

Return the Euler number of index n .

See [Wikipedia article Euler_number](#).

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import euler_number
sage: [euler_number(i) for i in range(8)]
[1, 0, -1, 0, 5, 0, -61, 0]

```

`sage.libs.flint.arith_sage.harmonic_number(n)`

Return the harmonic number H_n .

See [Wikipedia article Harmonic_number](#).

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import harmonic_number
sage: n = 500 + randint(0, 500)
sage: bool( sum(1/k for k in range(1, n+1)) == harmonic_number(n) )
True

```

`sage.libs.flint.arith_sage.number_of_partitions(n)`

Return the number of partitions of the integer n .

See [Wikipedia article Partition_\(number_theory\)](#).

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import number_of_partitions
sage: number_of_partitions(3)
3
sage: number_of_partitions(10)
42
sage: number_of_partitions(40)

```

(continues on next page)

(continued from previous page)

```

37338
sage: number_of_partitions(100)
190569292
sage: number_of_partitions(100000)
274935105697756965126775163209863526881734293159800547582031259843021473281149641730550507416607

```

`sage.libs.flint.arith_sage.stirling_number_1(n, k)`

Return the unsigned Stirling number of the first kind.

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import stirling_number_1
sage: [stirling_number_1(8,i) for i in range(9)]
[0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1]

```

`sage.libs.flint.arith_sage.stirling_number_2(n, k)`

Return the Stirling number of the second kind.

EXAMPLES:

```

sage: from sage.libs.flint.arith_sage import stirling_number_2
sage: [stirling_number_2(8,i) for i in range(9)]
[0, 1, 127, 966, 1701, 1050, 266, 28, 1]

```

3.4 Interface to FLINT's `qsieve_factor()`. This used to interact

with an external “QuadraticSieve” program, but its functionality has been absorbed into flint.

`sage.libs.flint.qsieve_sage.qsieve(n)`

Factor n using the quadratic sieve.

INPUT:

- n – integer; neither prime nor a perfect power

OUTPUT:

A list of the factors of n . There is no guarantee that the factors found will be prime, or distinct.

EXAMPLES:

```

sage: k = 19; n = next_prime(10^k)*next_prime(10^(k+1))
sage: factor(n) # (currently) uses PARI
10000000000000000000051 * 10000000000000000000039
sage: qsieve(n)
[(10000000000000000000051, 1), (10000000000000000000039, 1)]

```

3.5 File: `sage/libs/flint/ulong_extras_sage.pyx` (starting at line 1)

`sage.libs.flint.ulong_extras_sage.n_factor_to_list` (*n*, *proved*)

A wrapper around `n_factor`.

EXAMPLES:

```
sage: from sage.libs.flint.ulong_extras_sage import n_factor_to_list
sage: n_factor_to_list(60, 20)
[(2, 2), (3, 1), (5, 1)]
sage: n_factor_to_list((10**6).next_prime() + 1, 0)
[(2, 2), (53, 2), (89, 1)]
```


4.1 Wrappers for Giac functions

We provide a python function to compute and convert to sage a Groebner basis using the `giacpy_sage` module.

AUTHORS:

- Martin Albrecht (2015-07-01): initial version
- Han Frederic (2015-07-01): initial version

EXAMPLES:

```
sage: from sage.libs.giac import groebner_basis as gb_giac # random
sage: P = PolynomialRing(QQ, 6, 'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: B = gb_giac(I.gens()) # random
sage: B
Polynomial Sequence with 45 Polynomials in 6 Variables
```

```
class sage.libs.giac.GiacSettingsDefaultContext
```

Bases: object

Context preserve libgiac settings.

```
sage.libs.giac.groebner_basis(gens, proba_epsilon=None, threads=None, prot=False,
                             elim_variables=None, *args, **kwds)
```

Compute a Groebner Basis of an ideal using `giacpy_sage`. The result is automatically converted to sage.

Supported term orders of the underlying polynomial ring are `lex`, `deglex`, `degrevlex` and block orders with 2 `degrevlex` blocks.

INPUT:

- `gens` – an ideal (or a list) of polynomials over a prime field of characteristic 0 or $p < 2^31$
- `proba_epsilon` – (default: None) majoration of the probability of a wrong answer when probabilistic algorithms are allowed
 - if `proba_epsilon` is None, the value of `sage.structure.proof.all.polynomial()` is taken. If it is false then the global `giacpy_sage.giacsettings.proba_epsilon` is used.
 - if `proba_epsilon` is 0, probabilistic algorithms are disabled.
- `threads` – (default: None) maximal number of threads allowed for `giac`. If None, the global `giacpy_sage.giacsettings.threads` is considered.
- `prot` – boolean (default: False); if True print detailed information

- `elim_variables` – (default: `None`) a list of variables to eliminate from the ideal
 - if `elim_variables` is `None`, a Groebner basis with respect to the term ordering of the parent polynomial ring of the polynomials `gens` is computed.
 - if `elim_variables` is a list of variables, a Groebner basis of the elimination ideal with respect to a degrevlex term order is computed, regardless of the term order of the polynomial ring.

OUTPUT: polynomial sequence of the reduced Groebner basis

EXAMPLES:

```
sage: from sage.libs.giac import groebner_basis as gb_giac
sage: P = PolynomialRing(GF(previous_prime(2**31)), 6, 'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: B = gb_giac(I.gens())
...
sage: B
Polynomial Sequence with 45 Polynomials in 6 Variables
sage: B.is_groebner()
True
```

Elimination ideals can be computed by passing `elim_variables`:

```
sage: P = PolynomialRing(GF(previous_prime(2**31)), 5, 'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: B = gb_giac(I.gens(), elim_variables=[P.gen(0), P.gen(2)])
...
sage: B.is_groebner()
True
sage: B.ideal() == I.elimination_ideal([P.gen(0), P.gen(2)])
True
```

Computations over `QQ` can benefit from

- a probabilistic lifting:

```
sage: P = PolynomialRing(QQ, 5, 'x')
sage: I = ideal([P.random_element(3,7) for j in range(5)])
sage: B1 = gb_giac(I.gens(), 1e-16) # long time (1s)
...
sage: sage.structure.proof.all.polynomial(True)
sage: B2 = gb_giac(I.gens()) # long time (4s)
...
sage: B1 == B2 # long time
True
sage: B1.is_groebner() # not tested, too long time (50s)
True
```

- multi threaded operations:

```
sage: P = PolynomialRing(QQ, 8, 'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: time B = gb_giac(I.gens(), 1e-6, threads=2) # doctest: +SKIP
...
Time: CPU 168.98 s, Wall: 94.13 s
```

You can get detailed information by setting `prot=True`

```

sage: I = sage.rings.ideal.Katsura(P)
sage: gb_giac(I,prot=True) # random, long time (3s)
9381383 begin computing basis modulo 535718473
9381501 begin new iteration zmod, number of pairs: 8, base size: 8
...end, basis size 74 prime number 1
G=Vector [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,...
...creating reconstruction #0
...
+++++++basis size 74
checking pairs for i=0, j=
checking pairs for i=1, j=2,6,12,17,19,24,29,34,39,42,43,48,56,61,64,69,
...
checking pairs for i=72, j=73,
checking pairs for i=73, j=
Number of critical pairs to check 373
+++++++.....
Successful... check of 373 critical pairs
12380865 end final check
Polynomial Sequence with 74 Polynomials in 8 Variables

```

`sage.libs.giac.local_giacsettings` (*func*)

Decorator to preserve Giac's `proba_epsilon` and `threads` settings.

EXAMPLES:

```

sage: def testf(a, b):
.....:     giacsettings.proba_epsilon = a/100
.....:     giacsettings.threads = b+2
.....:     return (giacsettings.proba_epsilon, giacsettings.threads)

sage: from sage.libs.giac.giac import giacsettings
sage: from sage.libs.giac import local_giacsettings
sage: gporig, gtorig = (giacsettings.proba_epsilon, giacsettings.threads)
sage: gp, gt = local_giacsettings(testf)(giacsettings.proba_epsilon, giacsettings.
↳threads)
sage: gp == giacsettings.proba_epsilon
True
sage: gt == giacsettings.threads
True
sage: gp < gporig, gt - gtorig
(True, 2)

```


5.1 The Elliptic Curve Method for Integer Factorization (ECM)

Sage includes GMP-ECM, which is a highly optimized implementation of Lenstra's elliptic curve factorization method. See <https://gitlab.inria.fr/zimmerma/ecm> for more about GMP-ECM. This file provides a Cython interface to the GMP-ECM library.

AUTHORS:

- Robert L Miller (2008-01-21): library interface (clone of ecmfactor.c)
- Jeroen Demeyer (2012-03-29): signal handling, documentation
- Paul Zimmermann (2011-05-22) – added input/output of sigma

EXAMPLES:

```
sage: from sage.libs.libecm import ecmfactor
sage: result = ecmfactor(999, 0.00)
sage: result[0]
True
sage: result[1] in [3, 9, 27, 37, 111, 333, 999] or result[1]
True
sage: result = ecmfactor(999, 0.00, verbose=True)
Performing one curve with B1=0
Found factor in step 1: ...
sage: result[0]
True
sage: result[1] in [3, 9, 27, 37, 111, 333, 999] or result[1]
True
sage: ecmfactor(2^128+1, 1000, sigma=227140902)
(True, 5704689200685129054721, 227140902)
```

`sage.libs.libecm.ecmfactor` (*number*, *B1*, *verbose=False*, *sigma=0*)

Try to find a factor of a positive integer using ECM (Elliptic Curve Method). This function tries one elliptic curve.

INPUT:

- *number* – positive integer to be factored
- *B1* – bound for step 1 of ECM
- *verbose* – boolean (default: `False`); print some debugging information

OUTPUT:

Either `(False, None)` if no factor was found, or `(True, f)` if the factor *f* was found.

EXAMPLES:

```
sage: from sage.libs.libecm import ecmfactor
```

This number has a small factor which is easy to find for ECM:

```
sage: N = 2^167 - 1
sage: factor(N)
2349023 * 79638304766856507377778616296087448490695649
sage: ecmfactor(N, 2e5)
(True, 2349023, ...)
```

If a factor was found, we can reproduce the factorization with the same sigma value:

```
sage: N = 2^167 - 1
sage: ecmfactor(N, 2e5, sigma=1473308225)
(True, 2349023, 1473308225)
```

With a smaller B1 bound, we may or may not succeed:

```
sage: ecmfactor(N, 1e2) # random
(False, None)
```

The following number is a Mersenne prime, so we don't expect to find any factors (there is an extremely small chance that we get the input number back as factorization):

```
sage: N = 2^127 - 1
sage: N.is_prime()
True
sage: ecmfactor(N, 1e3)
(False, None)
```

If we have several small prime factors, it is possible to find a product of primes as factor:

```
sage: N = 2^179 - 1
sage: factor(N)
359 * 1433 * 1489459109360039866456940197095433721664951999121
sage: ecmfactor(N, 1e3) # random
(True, 514447, 3475102204)
```

We can ask for verbose output:

```
sage: N = 12^97 - 1
sage: factor(N)
11 * _
↪4357006235375344605345561005667974000505696611184208940783890278320995998159307781133050732832
sage: ecmfactor(N, 100, verbose=True)
Performing one curve with B1=100
Found factor in step 1: 11
(True, 11, ...)
sage: ecmfactor(N/11, 100, verbose=True)
Performing one curve with B1=100
Found no factor.
(False, None)
```

6.1 GSL arrays

class sage.libs.gsl.array.GSLDoubleArray

Bases: object

EXAMPLES:

```
sage: a = WaveletTransform(128, 'daubechies', 4)
sage: for i in range(1, 11):
....:     a[i] = 1
sage: a[:6:2]
[0.0, 1.0, 1.0]
```


7.1 Rubinstein's lcalc library

This is a wrapper around Michael Rubinstein's lcalc. See http://oto.math.uwaterloo.ca/~mrubinst/L_function_public/CODE/.

AUTHORS:

- Rishikesh (2010): added `compute_rank()` and `hardy_z_function()`
- Yann Laigle-Chapuy (2009): refactored
- Rishikesh (2009): initial version

class `sage.libs.lcalc.lcalc_Lfunction.Lfunction`

Bases: `object`

Initialization of L -function objects. See derived class for details, this class is not supposed to be instantiated directly.

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: Lfunction_from_character(DirichletGroup(5)[1])
L-function with complex Dirichlet coefficients
```

compute_rank()

Compute the analytic rank (the order of vanishing at the center) of the L -function.

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[2] # This is a quadratic character
sage: L = Lfunction_from_character(chi, type='int')
sage: L.compute_rank()
0
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: E = EllipticCurve([-82, 0])
sage: L = Lfunction_from_elliptic_curve(E, number_of_coeffs=40000)
sage: L.compute_rank()
3
```

find_zeros(T1, T2, stepsize)

Finds zeros on critical line between $T1$ and $T2$ using step size of `stepsize`. This function might miss zeros if step size is too large. This function computes the zeros of the L -function by using change in signs of a real valued function whose zeros coincide with the zeros of L -function.

Use `find_zeros_via_N()` for slower but more rigorous computation.

INPUT:

- T1 – a real number giving the lower bound
- T2 – a real number giving the upper bound
- stepsize – step size to be used for the zero search

OUTPUT: list of the imaginary parts of the zeros which were found

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[2] # This is a quadratic character
sage: L = Lfunction_from_character(chi, type='int')
sage: L.find_zeros(5,15,.1)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
sage: L = Lfunction_from_character(chi, type='double')
sage: L.find_zeros(1,15,.1)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[1]
sage: L = Lfunction_from_character(chi, type='complex')
sage: L.find_zeros(-8,8,.1)
[-4.13290370521..., 6.18357819545...]
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: L = Lfunction_Zeta()
sage: L.find_zeros(10,29.1,.1)
[14.1347251417..., 21.0220396387..., 25.0108575801...]
```

find_zeros_via_N (*count=0, start=0, max_refine=1025, rank=-1*)

Find *count* zeros (in order of increasing magnitude) and output their imaginary parts. This function verifies that no zeros are missed, and that all values output are indeed zeros.

If this *L*-function is self-dual (if its Dirichlet coefficients are real, up to a tolerance of $1e-6$), then only the zeros with positive imaginary parts are output. Their conjugates, which are also zeros, are not output.

INPUT:

- *count* – number of zeros to be found
- *start* – (default: 0) how many initial zeros to skip
- *max_refine* – when some zeros are found to be missing, the step size used to find zeros is refined. *max_refine* gives an upper limit on when *lcalc* should give up. Use default value unless you know what you are doing.
- *rank* – integer (default: -1); analytic rank of the *L*-function. If -1 is passed, then we attempt to compute it. (Use default if in doubt)

OUTPUT: list of the imaginary parts of the zeros that have been found

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[2] #This is a quadratic character
sage: L = Lfunction_from_character(chi, type='int')
```

(continues on next page)

(continued from previous page)

```
sage: L.find_zeros_via_N(3)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
sage: L = Lfunction_from_character(chi, type='double')
sage: L.find_zeros_via_N(3)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[1]
sage: L = Lfunction_from_character(chi, type='complex')
sage: zeros = L.find_zeros_via_N(3)
sage: (zeros[0] - (-4.13290370521286)).abs() < 1e-8
True
sage: (zeros[1] - 6.18357819545086).abs() < 1e-8
True
sage: (zeros[2] - 8.45722917442320).abs() < 1e-8
True
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: L = Lfunction_Zeta()
sage: L.find_zeros_via_N(3)
[14.1347251417..., 21.0220396387..., 25.0108575801...]
```

hardy_z_function(*s*)

Compute the Hardy Z-function of the L -function at s .

INPUT:

- s – a complex number with imaginary part between -0.5 and 0.5

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[2] # Quadratic character
sage: L = Lfunction_from_character(chi, type='int')
sage: (L.hardy_z_function(0) - 0.231750947504).abs() < 1e-8
True
sage: L.hardy_z_function(0.5).imag().abs() < 1e-8
True
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[1]
sage: L = Lfunction_from_character(chi, type='complex')
sage: (L.hardy_z_function(0) - 0.793967590477).abs() < 1e-8
True
sage: L.hardy_z_function(0.5).imag().abs() < 1e-8
True
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: E = EllipticCurve([-82, 0])
sage: L = Lfunction_from_elliptic_curve(E, number_of_coeffs=40000)
sage: (L.hardy_z_function(2.1) - (-0.006431791768)).abs() < 1e-8
True
```

value(*s*, *derivative*=0)

Compute the value of the L -function at s .

INPUT:

- `s` – a complex number
- `derivative` – integer (default: 0); the derivative to be evaluated
- `rotate` – boolean (default: False); if True, this returns the value of the Hardy Z-function (sometimes called the Riemann-Siegel Z-function or the Siegel Z-function)

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[2] # This is a quadratic character
sage: L = Lfunction_from_character(chi, type='int')
sage: (L.value(0.5) - 0.231750947504016).abs() < 1e-8
True
sage: v = L.value(0.2 + 0.4*I)
sage: (v - (0.102558603193 + 0.190840777924*I)).abs() < 1e-8
True
sage: L = Lfunction_from_character(chi, type='double')
sage: (L.value(0.6) - 0.274633355856345).abs() < 1e-8
True
sage: v = L.value(0.6 + I)
sage: (v - (0.362258705721 + 0.43388825062*I)).abs() < 1e-8
True
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi = DirichletGroup(5)[1]
sage: L = Lfunction_from_character(chi, type='complex')
sage: v = L.value(0.5)
sage: (v - (0.763747880117 + 0.21696476751*I)).abs() < 1e-8
True
sage: v = L.value(0.6 + 5*I)
sage: (v - (0.702723260619 - 1.10178575243*I)).abs() < 1e-8
True
```

```
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: L = Lfunction_Zeta()
sage: (L.value(0.5) + 1.46035450880).abs() < 1e-8
True
sage: v = L.value(0.4 + 0.5*I)
sage: (v - (-0.450728958517 - 0.780511403019*I)).abs() < 1e-8
True
```

class `sage.libs.lcalc.lcalc_Lfunction.Lfunction_C`

Bases: `Lfunction`

The `Lfunction_C` class is used to represent L -functions with complex Dirichlet Coefficients. We assume that L -functions satisfy the following functional equation.

$$\Lambda(s) = \omega Q^s \overline{\Lambda(1 - \bar{s})}$$

where

$$\Lambda(s) = Q^s \left(\prod_{j=1}^a \Gamma(\kappa_j s + \gamma_j) \right) L(s)$$

See (23) in [arXiv math/0412181](https://arxiv.org/abs/math/0412181)

INPUT:

- `what_type_L` – integer; this should be set to 1 if the coefficients are periodic and 0 otherwise
- `dirichlet_coefficient` – list of Dirichlet coefficients of the L -function. Only first M coefficients are needed if they are periodic.
- `period` – if the coefficients are periodic, this should be the period of the coefficients
- `Q` – see above
- `OMEGA` – see above
- `kappa` – list of the values of κ_j in the functional equation
- `gamma` – list of the values of γ_j in the functional equation
- `pole` – list of the poles of L -function
- `residue` – list of the residues of the L -function

Note

If an L -function satisfies $\Lambda(s) = \omega Q^s \Lambda(k - s)$, by replacing s by $s + (k - 1)/2$, one can get it in the form we need.

class `sage.libs.lcalc.lcalc_Lfunction.Lfunction_D`

Bases: `Lfunction`

The `Lfunction_D` class is used to represent L -functions with real Dirichlet coefficients. We assume that L -functions satisfy the following functional equation.

$$\Lambda(s) = \omega Q^s \overline{\Lambda(1 - \bar{s})}$$

where

$$\Lambda(s) = Q^s \left(\prod_{j=1}^a \Gamma(\kappa_j s + \gamma_j) \right) L(s)$$

See (23) in [arXiv math/0412181](https://arxiv.org/abs/math/0412181)

INPUT:

- `what_type_L` – integer; this should be set to 1 if the coefficients are periodic and 0 otherwise
- `dirichlet_coefficient` – list of Dirichlet coefficients of the L -function. Only first M coefficients are needed if they are periodic.
- `period` – if the coefficients are periodic, this should be the period of the coefficients
- `Q` – see above
- `OMEGA` – see above
- `kappa` – list of the values of κ_j in the functional equation
- `gamma` – list of the values of γ_j in the functional equation
- `pole` – list of the poles of L -function
- `residue` – list of the residues of the L -function

Note

If an L -function satisfies $\Lambda(s) = \omega Q^s \Lambda(k - s)$, by replacing s by $s + (k - 1)/2$, one can get it in the form we need.

class `sage.libs.lcalc.lcalc_Lfunction.Lfunction_I`

Bases: `Lfunction`

The `Lfunction_I` class is used to represent L -functions with integer Dirichlet Coefficients. We assume that L -functions satisfy the following functional equation.

$$\Lambda(s) = \omega Q^s \overline{\Lambda(1 - \bar{s})}$$

where

$$\Lambda(s) = Q^s \left(\prod_{j=1}^a \Gamma(\kappa_j s + \gamma_j) \right) L(s)$$

See (23) in [arXiv math/0412181](https://arxiv.org/abs/math/0412181)

INPUT:

- `what_type_L` – integer; this should be set to 1 if the coefficients are periodic and 0 otherwise
- `dirichlet_coefficient` – list of Dirichlet coefficients of the L -function. Only first M coefficients are needed if they are periodic.
- `period` – if the coefficients are periodic, this should be the period of the coefficients
- `Q` – see above
- `OMEGA` – see above
- `kappa` – list of the values of κ_j in the functional equation
- `gamma` – list of the values of γ_j in the functional equation
- `pole` – list of the poles of L -function
- `residue` – list of the residues of the L -function

Note

If an L -function satisfies $\Lambda(s) = \omega Q^s \Lambda(k - s)$, by replacing s by $s + (k - 1)/2$, one can get it in the form we need.

class `sage.libs.lcalc.lcalc_Lfunction.Lfunction_Zeta`

Bases: `Lfunction`

The `Lfunction_Zeta` class is used to generate the Riemann zeta function.

`sage.libs.lcalc.lcalc_Lfunction.Lfunction_from_character` (*chi*, *type='complex'*)

Given a primitive Dirichlet character, this function returns an `lcalc` L -function object for the L -function of the character.

INPUT:

- `chi` – a Dirichlet character

- `use_type` – string (default: 'complex'); type used for the Dirichlet coefficients. This can be 'int', 'double' or 'complex'.

OUTPUT: L -function object for χ

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import Lfunction_from_character
sage: Lfunction_from_character(DirichletGroup(5)[1])
L-function with complex Dirichlet coefficients
sage: Lfunction_from_character(DirichletGroup(5)[2], type='int')
L-function with integer Dirichlet coefficients
sage: Lfunction_from_character(DirichletGroup(5)[2], type='double')
L-function with real Dirichlet coefficients
sage: Lfunction_from_character(DirichletGroup(5)[1], type='int')
Traceback (most recent call last):
...
ValueError: For non quadratic characters you must use type="complex"
```

`sage.libs.lcalc.lcalc_Lfunction.Lfunction_from_elliptic_curve` (E , *number_of_coeffs*=10000)

Given an elliptic curve E , return an L -function object for the function $L(s, E)$.

INPUT:

- E – an elliptic curve
- `number_of_coeffs` – integer (default: 10000); the number of coefficients to be used when constructing the L -function object. Right now this is fixed at object creation time, and is not automatically set intelligently.

OUTPUT: L -function object for $L(s, E)$

EXAMPLES:

```
sage: from sage.libs.lcalc.lcalc_Lfunction import Lfunction_from_elliptic_curve
sage: L = Lfunction_from_elliptic_curve(EllipticCurve('37'))
sage: L
L-function with real Dirichlet coefficients
sage: L.value(0.5).abs() < 1e-8
True
sage: (L.value(0.5, derivative=1) - 0.305999773835200).abs() < 1e-6
True
```


LIBSINGULAR

8.1 libSingular: Functions

Sage implements a C wrapper around the Singular interpreter which allows to call any function directly from Sage without string parsing or interprocess communication overhead. Users who do not want to call Singular functions directly, usually do not have to worry about this interface, since it is handled by higher level functions in Sage.

EXAMPLES:

The direct approach for loading a Singular function is to call the function `singular_function()` with the function name as parameter:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<a,b,c,d> = PolynomialRing(GF(7))
sage: std = singular_function('std')
sage: I = sage.rings.ideal.Cyclic(P)
sage: std(I)
[a + b + c + d,
 b^2 + 2*b*d + d^2,
 b*c^2 + c^2*d - b*d^2 - d^3,
 b*c*d^2 + c^2*d^2 - b*d^3 + c*d^3 - d^4 - 1,
 b*d^4 + d^5 - b - d,
 c^3*d^2 + c^2*d^3 - c - d,
 c^2*d^4 + b*c - b*d + c*d - 2*d^2]
```

If a Singular library needs to be loaded before a certain function is available, use the `lib()` function as shown below:

```
sage: from sage.libs.singular.function import singular_function, lib as singular_lib
sage: primdecSY = singular_function('primdecSY')
Traceback (most recent call last):
...
NameError: Singular library function 'primdecSY' is not defined

sage: singular_lib('primdec.lib')
sage: primdecSY = singular_function('primdecSY')
```

There is also a short-hand notation for the above:

```
sage: import sage.libs.singular.function_factory
sage: primdecSY = sage.libs.singular.function_factory.ff.primdec__lib.primdecSY
```

The above line will load “primdec.lib” first and then load the function `primdecSY`.

AUTHORS:

- Michael Brickenstein (2009-07): initial implementation, overall design

- Martin Albrecht (2009-07): clean up, enhancements, etc.
- Michael Brickenstein (2009-10): extension to more Singular types
- Martin Albrecht (2010-01): clean up, support for attributes
- Simon King (2011-04): include the documentation provided by Singular as a code block
- Burcin Erocal, Michael Brickenstein, Oleksandr Motsak, Alexander Dreyer, Simon King (2011-09): plural support

class `sage.libs.singular.function.BaseCallHandler`

Bases: `object`

A call handler is an abstraction which hides the details of the implementation differences between kernel and library functions.

class `sage.libs.singular.function.Converter`

Bases: `SageObject`

A *Converter* interfaces between Sage objects and Singular interpreter objects.

ring()

Return the ring in which the arguments of this list live.

EXAMPLES:

```
sage: from sage.libs.singular.function import Converter
sage: P.<a,b,c> = PolynomialRing(GF(127))
sage: Converter([a,b,c],ring=P).ring()
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127
```

class `sage.libs.singular.function.KernelCallHandler`

Bases: `BaseCallHandler`

A call handler is an abstraction which hides the details of the implementation differences between kernel and library functions.

This class implements calling a kernel function.

Note

Do not construct this class directly, use `singular_function()` instead.

class `sage.libs.singular.function.LibraryCallHandler`

Bases: `BaseCallHandler`

A call handler is an abstraction which hides the details of the implementation differences between kernel and library functions.

This class implements calling a library function.

Note

Do not construct this class directly, use `singular_function()` instead.

class `sage.libs.singular.function.Resolution`

Bases: `object`

A simple wrapper around Singular's resolutions.

class sage.libs.singular.function.RingWrap

Bases: object

A simple wrapper around Singular's rings.

characteristic()

Get characteristic.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).characteristic()
0
```

is_commutative()

Determine whether a given ring is commutative.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).is_commutative()
True
```

ngens()

Get number of generators.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).ngens()
3
```

npars()

Get number of parameters.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).npars()
0
```

ordering_string()

Get Singular string defining monomial ordering.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).ordering_string()
'dp(3),C'
```

par_names()

Get parameter names.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).par_names()
[]
```

var_names()

Get names of variables.

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: ringlist = singular_function("ringlist")
sage: l = ringlist(P)
sage: ring = singular_function("ring")
sage: ring(l, ring=P).var_names()
['x', 'y', 'z']
```

class sage.libs.singular.function.SingularFunction

Bases: *SageObject*

The base class for Singular functions either from the kernel or from the library.

class sage.libs.singular.function.SingularKernelFunction

Bases: *SingularFunction*

EXAMPLES:

```
sage: from sage.libs.singular.function import SingularKernelFunction
sage: R.<x,y> = PolynomialRing(QQ, order='lex')
sage: I = R.ideal(x, x+1)
sage: f = SingularKernelFunction("std")
sage: f(I)
[1]
```

class sage.libs.singular.function.SingularLibraryFunction

Bases: *SingularFunction*

EXAMPLES:

```

sage: from sage.libs.singular.function import SingularLibraryFunction
sage: R.<x,y> = PolynomialRing(QQ, order='lex')
sage: I = R.ideal(x, x+1)
sage: f = SingularLibraryFunction("groebner")
sage: f(I)
[1]

```

`sage.libs.singular.function.all_singular_poly_wrapper(s)`

Test for a sequence `s`, whether it consists of singular polynomials.

EXAMPLES:

```

sage: from sage.libs.singular.function import all_singular_poly_wrapper
sage: P.<x,y,z> = QQ[]
sage: all_singular_poly_wrapper([x+1, y])
True
sage: all_singular_poly_wrapper([x+1, y, 1])
False

```

`sage.libs.singular.function.all_vectors(s)`

Check if a sequence `s` consists of free module elements over a singular ring.

EXAMPLES:

```

sage: from sage.libs.singular.function import all_vectors
sage: P.<x,y,z> = QQ[]
sage: M = P**2
sage: all_vectors([x])
False
sage: all_vectors([(x,y)])
False
sage: all_vectors([M(0), M((x,y))])
True
sage: all_vectors([M(0), M((x,y)), (0,0)])
False

```

`sage.libs.singular.function.get_printlevel()`

Return the value of the variable `printlevel`.

This is useful to switch off and back the comments.

EXAMPLES:

```

sage: from sage.libs.singular.function import get_printlevel, set_printlevel
sage: l = get_printlevel()
sage: set_printlevel(-1)
sage: get_printlevel()
-1
sage: set_printlevel(1)

```

`sage.libs.singular.function.is_sage_wrapper_for_singular_ring(ring)`

Check whether wrapped ring arises from Singular or Singular/Plural.

EXAMPLES:

```

sage: from sage.libs.singular.function import is_sage_wrapper_for_singular_ring
sage: P.<x,y,z> = QQ[]

```

(continues on next page)

(continued from previous page)

```
sage: is_sage_wrapper_for_singular_ring(P)
True
```

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')
sage: is_sage_wrapper_for_singular_ring(P)
True
```

`sage.libs.singular.function.is_singular_poly_wrapper(p)`

Check if `p` is some data type corresponding to some singular poly.

EXAMPLES:

```
sage: from sage.libs.singular.function import is_singular_poly_wrapper
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({z*x:x*z+2*x, z*y:y*z-2*y})
sage: is_singular_poly_wrapper(x+y)
True
```

`sage.libs.singular.function.lib(name)`

Load the Singular library name.

INPUT:

- `name` – a Singular library name

EXAMPLES:

```
sage: from sage.libs.singular.function import singular_function
sage: from sage.libs.singular.function import lib as singular_lib
sage: singular_lib('general.lib')
sage: primes = singular_function('primes')
sage: primes(2,10, ring=GF(127)['x,y,z'])
(2, 3, 5, 7)
```

`sage.libs.singular.function.list_of_functions(packages=False)`

Return a list of all function names currently available.

INPUT:

- `packages` – include local functions in packages

EXAMPLES:

```
sage: from sage.libs.singular.function import list_of_functions
sage: 'groebner' in list_of_functions()
True
```

`sage.libs.singular.function.set_printlevel(l)`

Set the value of the variable `printlevel`.

This is useful to switch off and back the comments.

EXAMPLES:

```
sage: from sage.libs.singular.function import get_printlevel, set_printlevel
sage: l = get_printlevel()
sage: set_printlevel(2)
```

(continues on next page)

(continued from previous page)

```
sage: get_printlevel()
2
sage: set_printlevel(1)
```

sage.libs.singular.function.**singular_function**(*name*)

Construct a new libSingular function object for the given name.

This function works both for interpreter and built-in functions.

INPUT:

- name – the name of the function

EXAMPLES:

```
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f = 3*x*y + 2*z + 1
sage: g = 2*x + 1/2
sage: I = Ideal([f,g])
```

```
sage: from sage.libs.singular.function import singular_function
sage: std = singular_function("std")
sage: std(I)
[3*y - 8*z - 4, 4*x + 1]
sage: size = singular_function("size")
sage: size([2, 3, 3])
3
sage: size("sage")
4
sage: size(["hello", "sage"])
2
sage: factorize = singular_function("factorize")
sage: factorize(f)
[[1, 3*x*y + 2*z + 1], (1, 1)]
sage: factorize(f, 1)
[3*x*y + 2*z + 1]
```

We give a wrong number of arguments:

```
sage: factorize()
Traceback (most recent call last):
...
RuntimeError: error in Singular function call 'factorize':
Wrong number of arguments (got 0 arguments, arity is CMD_12)
sage: factorize(f, 1, 2)
Traceback (most recent call last):
...
RuntimeError: error in Singular function call 'factorize':
Wrong number of arguments (got 3 arguments, arity is CMD_12)
sage: factorize(f, 1, 2, 3)
Traceback (most recent call last):
...
RuntimeError: error in Singular function call 'factorize':
Wrong number of arguments (got 4 arguments, arity is CMD_12)
```

The Singular function list can be called with any number of arguments:

```

sage: singular_list = singular_function("list")
sage: singular_list(2, 3, 6)
[2, 3, 6]
sage: singular_list()
[]
sage: singular_list(1)
[1]
sage: singular_list(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

```

We try to define a non-existing function:

```

sage: number_foobar = singular_function('number_foobar')
Traceback (most recent call last):
...
NameError: Singular library function 'number_foobar' is not defined

```

```

sage: from sage.libs.singular.function import lib as singular_lib
sage: singular_lib('general.lib')
sage: number_e = singular_function('number_e')
sage: number_e(10r)
67957045707/25000000000
sage: RR(number_e(10r))
2.71828182828000

```

```

sage: singular_lib('primdec.lib')
sage: primdecGTZ = singular_function("primdecGTZ")
sage: primdecGTZ(I)
[[[y - 8/3*z - 4/3, x + 1/4], [y - 8/3*z - 4/3, x + 1/4]]]
sage: singular_list((1,2,3),3,[1,2,3], ring=P)
[(1, 2, 3), 3, [1, 2, 3]]
sage: ringlist=singular_function("ringlist")
sage: l = ringlist(P)
sage: l[3].__class__
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic
↳'>
sage: l
[0, ['x', 'y', 'z'], [['dp', (1, 1, 1)], ['C', (0,) ]], [0]]
sage: ring=singular_function("ring")
sage: ring(l)
<RingWrap>
sage: matrix = Matrix(P,2,2)
sage: matrix.randomize(terms=1)
sage: det = singular_function("det")
sage: det(matrix) == matrix[0, 0] * matrix[1, 1] - matrix[0, 1] * matrix[1, 0]
True
sage: coeffs = singular_function("coeffs")
sage: coeffs(x*y+y+1,y)
[ 1]
[x + 1]
sage: intmat = Matrix(ZZ, 2,2, [100,2,3,4])
sage: det(intmat)
394
sage: random = singular_function("random")
sage: A = random(10,2,3); A.nrows(), max(A.list()) <= 10
(2, True)
sage: P.<x,y,z> = PolynomialRing(QQ)

```

(continues on next page)

(continued from previous page)

```

sage: M=P**3
sage: leadcoef = singular_function("leadcoef")
sage: v=M((100*x,5*y,10*z*x*y))
sage: leadcoef(v)
10
sage: v = M([x+y,x*y+y**3,z])
sage: lead = singular_function("lead")
sage: lead(v)
(0, y^3)
sage: jet = singular_function("jet")
sage: jet(v, 2)
(x + y, x*y, z)
sage: syz = singular_function("syz")
sage: I = P.ideal([x+y,x*y-y, y*2,x**2+1])
sage: M = syz(I)
sage: M
[(-2*y, 2, y + 1, 0), (0, -2, x - 1, 0), (x*y - y, -y + 1, 1, -y), (x^2 + 1, -x -
↪ 1, -1, -x)]
sage: singular_lib("mprimdec.lib")
sage: syz(M)
[(-x - 1, y - 1, 2*x, -2*y)]
sage: GTZmod = singular_function("GTZmod")
sage: GTZmod(M)
[[[(-2*y, 2, y + 1, 0), (0, x + 1, 1, -y), (0, -2, x - 1, 0), (x*y - y, -y + 1, 1,
↪ -y), (x^2 + 1, 0, 0, -x - y)], [0]]]
sage: mres = singular_function("mres")
sage: resolution = mres(M, 0)
sage: resolution
<Resolution>
sage: singular_list(resolution)
[[(-2*y, 2, y + 1, 0), (0, -2, x - 1, 0), (x*y - y, -y + 1, 1, -y), (x^2 + 1, -x -
↪ 1, -1, -x)], [(-x - 1, y - 1, 2*x, -2*y)], [(0)]]

sage: A.<x,y> = FreeAlgebra(QQ, 2)
sage: P.<x,y> = A.g_algebra({y*x:-x*y})
sage: I= Sequence([x*y,x+y], check=False, immutable=True)
sage: twostd = singular_function("twostd")
sage: twostd(I)
[x + y, y^2]
sage: M=syz(I)
doctest...
sage: M
[(x + y, x*y)]
sage: syz(M)
[(0)]
sage: mres(I, 0)
<Resolution>
sage: M=P**3
sage: v=M((100*x,5*y,10*y*x*y))
sage: leadcoef(v)
-10
sage: v = M([x+y,x*y+y**3,x])
sage: lead(v)
(0, y^3)
sage: jet(v, 2)
(x + y, x*y, x)
sage: l = ringlist(P)

```

(continues on next page)

(continued from previous page)

```

sage: len(l)
6
sage: ring(l)
<noncommutative RingWrap>
sage: I=twostd(I)
sage: l[3]=I
sage: ring(l)
<noncommutative RingWrap>

```

8.2 libSingular: Function Factory

AUTHORS:

- Martin Albrecht (2010-01): initial version

class sage.libs.singular.function_factory.**SingularFunctionFactory**

Bases: object

A convenient interface to libsingular functions.

8.3 libSingular: Conversion Routines and Initialisation

AUTHOR:

- Martin Albrecht <malb@informatik.uni-bremen.de>
- Miguel Marco <mmarco@unizar.es> (2021): added transcendental extensions over Q

sage.libs.singular.singular.**get_resource**(*id*)

Return a Singular “resource”.

INPUT:

- *id* – a single-character string; see <https://github.com/Singular/Singular/blob/spielwiese/resources/feResource.cc>

OUTPUT: string or None

EXAMPLES:

```

sage: from sage.libs.singular.singular import get_resource
sage: get_resource('D')           # SINGULAR_DATA_DIR
'...'
sage: get_resource('i')           # SINGULAR_INFO_FILE
'.../singular...'
sage: get_resource('7') is None   # not defined
True

```

sage.libs.singular.singular.**si2sa_resolution**(*res*)

Pull the data from Singular resolution *res* to construct a Sage resolution.

INPUT:

- *res* – Singular resolution

The procedure is destructive and `res` is not usable afterward.

EXAMPLES:

```
sage: from sage.libs.singular.singular import si2sa_resolution
sage: from sage.libs.singular.function import singular_function
sage: module = singular_function("module")
sage: mres = singular_function('mres')

sage: S.<x,y,z,w> = PolynomialRing(QQ)
sage: I = S.ideal([y*w - z^2, -x*w + y*z, x*z - y^2])
sage: mod = module(I)
sage: r = mres(mod, 0)
sage: si2sa_resolution(r)
[
                                [ y  x]
                                [-z -y]
[z^2 - y*w y*z - x*w y^2 - x*z], [ w  z]
]
```

`sage.libs.singular.singular.si2sa_resolution_graded(res, degrees)`

Pull the data from Singular resolution `res` to construct a Sage resolution.

INPUT:

- `res` – Singular resolution
- `degrees` – list of integers or integer vectors

The procedure is destructive, and `res` is not usable afterward.

EXAMPLES:

```
sage: from sage.libs.singular.singular import si2sa_resolution_graded
sage: from sage.libs.singular.function import singular_function
sage: module = singular_function("module")
sage: mres = singular_function('mres')

sage: S.<x,y,z,w> = PolynomialRing(QQ)
sage: I = S.ideal([y*w - z^2, -x*w + y*z, x*z - y^2])
sage: mod = module(I)
sage: r = mres(mod, 0)
sage: res_mats, res_degs = si2sa_resolution_graded(r, (1, 1, 1, 1))
sage: res_mats
[
                                [ y  x]
                                [-z -y]
[z^2 - y*w y*z - x*w y^2 - x*z], [ w  z]
]
sage: res_degs
[[[2], [2], [2]], [[1, 1, 1], [1, 1, 1]]]
```

8.4 Wrapper for Singular's Polynomial Arithmetic

AUTHOR:

- Martin Albrecht (2009-07): refactoring

8.5 libSingular: Options

Singular uses a set of global options to determine verbosity and the behavior of certain algorithms. We provide an interface to these options in the most 'natural' python-ic way. Users who do not wish to deal with Singular functions directly usually do not have to worry about this interface or Singular options in general since this is taken care of by higher level functions.

We compute a Groebner basis for Cyclic-5 in two different contexts:

```
sage: P.<a,b,c,d,e> = PolynomialRing(GF(127))
sage: I = sage.rings.ideal.Cyclic(P)
sage: import sage.libs.singular.function_factory
sage: std = sage.libs.singular.function_factory.ff.std
```

By default, tail reductions are performed:

```
sage: from sage.libs.singular.option import opt, opt_ctx
sage: opt['red_tail']
True
sage: red = std(I)[-1]; red
d^2*e^6 + 28*b*c*d + ...
```

If we don't want this, we can create an option context, which disables this:

```
sage: with opt_ctx(red_tail=False, red_sb=False):
....:     notred = std(I)[-1]; notred
d^2*e^6 + ...
sage: len(list(red)) < len(list(notred))
True
```

However, this does not affect the global state:

```
sage: opt['red_tail']
True
```

On the other hand, any assignment to an option object will immediately change the global state:

```
sage: opt['red_tail'] = False
sage: opt['red_tail']
False
sage: opt['red_tail'] = True
sage: opt['red_tail']
True
```

Assigning values within an option context, only affects this context:

```
sage: with opt_ctx:
....:     opt['red_tail'] = False

sage: opt['red_tail']
True
```

Option contexts can also be safely stacked:

```
sage: with opt_ctx:
.....:     opt['red_tail'] = False
.....:     print(opt)
.....:     with opt_ctx:
.....:         opt['red_through'] = False
.....:         print(opt)
general options for libSingular (current value 0x00000082)
general options for libSingular (current value 0x00000002)

sage: print(opt)
general options for libSingular (current value 0x02000082)
```

Furthermore, the integer valued options `deg_bound` and `mult_bound` can be used:

```
sage: R.<x,y> = QQ[]
sage: I = R*[x^3+y^2,x^2*y+1]
sage: opt['deg_bound'] = 2
sage: std(I)
[x^2*y + 1, x^3 + y^2]
sage: opt['deg_bound'] = 0
sage: std(I)
[y^3 - x, x^2*y + 1, x^3 + y^2]
```

The same interface is available for verbosity options:

```
sage: from sage.libs.singular.option import opt_verb
sage: opt_verb['not_warn_sb']
False
sage: opt.reset_default() # needed to avoid side effects
sage: opt_verb.reset_default() # needed to avoid side effects
```

AUTHOR:

- Martin Albrecht (2009-08): initial implementation
- Martin Albrecht (2010-01): better interface, verbosity options
- Simon King (2010-07): Python-ic option names; `deg_bound` and `mult_bound`

class `sage.libs.singular.option.LibSingularOptions`

Bases: `LibSingularOptions_abstract`

Pythonic Interface to libSingular's options.

Supported options are:

- `return_sb` or `returnSB` – the functions `syz`, `intersect`, `quotient`, `modulo` return a standard base instead of a generating set if `return_sb` is set. This option should not be used for `lift`.
- `fast_hc` or `fastHC` – tries to find the highest corner of the staircase (HC) as fast as possible during a standard basis computation (only used for local orderings).
- `int_strategy` or `intStrategy` – avoids division of coefficients during standard basis computations. This option is ring dependent. By default, it is set for rings with characteristic 0 and not set for all other rings.
- `lazy` – uses a more lazy approach in `std` computations, which was used in SINGULAR version before 2-0 (and which may lead to faster or slower computations, depending on the example).
- `length` – select shorter reducers in `std` computations

- `not_regularity` or `notRegularity` – disables the regularity bound for `res` and `mres`
- `not_sugar` or `notSugar` – disables the sugar strategy during standard basis computation
- `not_buckets` or `notBuckets` – disables the bucket representation of polynomials during standard basis computations. This option usually decreases the memory usage but increases the computation time. It should only be set for memory-critical standard basis computations.
- `old_std` or `oldStd` – uses a more lazy approach in `std` computations, which was used in SINGULAR version before 2-0 (and which may lead to faster or slower computations, depending on the example).
- `prot` – shows protocol information indicating the progress during the following computations: `facstd`, `fglm`, `groebner`, `lres`, `mres`, `minres`, `mstd`, `res`, `slimgb`, `sres`, `std`, `stdfglm`, `stdhilb`, `syz`.
- `red_sb` or `redSB` – computes a reduced standard basis in any standard basis computation
- `red_tail` or `redTail` – reduction of the tails of polynomials during standard basis computations. This option is ring dependent. By default, it is set for rings with global degree orderings and not set for all other rings.
- `red_through` or `redThrough` – for inhomogeneous input, polynomial reductions during standard basis computations are never postponed, but always finished through. This option is ring dependent. By default, it is set for rings with global degree orderings and not set for all other rings.
- `sugar_crit` or `sugarCrit` – uses criteria similar to the homogeneous case to keep more useless pairs
- `weight_m` or `weightM` – automatically computes suitable weights for the weighted ecart and the weighted sugar method

In addition, two integer valued parameters are supported, namely:

- `deg_bound` or `degBound` – the standard basis computation is stopped if the total (weighted) degree exceeds `deg_bound`. `deg_bound` should not be used for a global ordering with inhomogeneous input. Reset this bound by setting `deg_bound` to 0. The exact meaning of “degree” depends on the ring ordering and the command: `slimgb` uses always the total degree with weights 1, `std` does so for block orderings, only.
- `mult_bound` or `multBound` – the standard basis computation is stopped if the ideal is zero-dimensional in a ring with local ordering and its multiplicity is lower than `mult_bound`. Reset this bound by setting `mult_bound` to 0.

EXAMPLES:

```
sage: from sage.libs.singular.option import LibSingularOptions
sage: libsingular_options = LibSingularOptions()
sage: libsingular_options
general options for libSingular (current value 0x06000082)
```

Here we demonstrate the intended way of using `libSingular` options:

```
sage: R.<x,y> = QQ[]
sage: I = R*[x^3+y^2, x^2*y+1]
sage: I.groebner_basis(deg_bound=2)
[x^3 + y^2, x^2*y + 1]
sage: I.groebner_basis()
[x^3 + y^2, x^2*y + 1, y^3 - x]
```

The option `mult_bound` is only relevant in the local case:

```
sage: from sage.libs.singular.option import opt
sage: Rlocal.<x,y,z> = PolynomialRing(QQ, order='ds')
```

(continues on next page)

(continued from previous page)

```

sage: x^2<x
True
sage: J = [x^7+y^7+z^6,x^6+y^8+z^7,x^7+y^5+z^8, x^2*y^3+y^2*z^3+x^3*z^2,x^3*y^2+y^
↳3*z^2+x^2*z^3]*Rlocal
sage: J.groebner_basis(mult_bound=100)
[x^3*y^2 + y^3*z^2 + x^2*z^3, x^2*y^3 + x^3*z^2 + y^2*z^3, y^5, x^6 + x*y^4*z^5,
↳x^4*z^2 - y^4*z^2 - x^2*y*z^3 + x*y^2*z^3, z^6 - x*y^4*z^4 - x^3*y*z^5]
sage: opt['red_tail'] = True # the previous commands reset opt['red_tail'] to
↳False
sage: J.groebner_basis()
[x^3*y^2 + y^3*z^2 + x^2*z^3, x^2*y^3 + x^3*z^2 + y^2*z^3, y^5, x^6, x^4*z^2 - y^
↳4*z^2 - x^2*y*z^3 + x*y^2*z^3, z^6, y^4*z^3 - y^3*z^4 - x^2*z^5, x^3*y*z^4 - x^
↳2*y^2*z^4 + x*y^3*z^4, x^3*z^5, x^2*y*z^5 + y^3*z^5, x*y^3*z^5]

```

reset_default()

Reset libSingular's default options.

EXAMPLES:

```

sage: from sage.libs.singular.option import opt
sage: opt['red_tail']
True
sage: opt['red_tail'] = False
sage: opt['red_tail']
False
sage: opt['deg_bound']
0
sage: opt['deg_bound'] = 2
sage: opt['deg_bound']
2
sage: opt.reset_default()
sage: opt['red_tail']
True
sage: opt['deg_bound']
0

```

class sage.libs.singular.option.LibSingularOptionsContext

Bases: object

Option context

This object localizes changes to options.

EXAMPLES:

```

sage: from sage.libs.singular.option import opt, opt_ctx
sage: opt
general options for libSingular (current value 0x06000082)

```

```

sage: with opt_ctx(redTail=False):
.....:     print(opt)
.....:     with opt_ctx(redThrough=False):
.....:         print(opt)
general options for libSingular (current value 0x04000082)
general options for libSingular (current value 0x04000002)

```

(continues on next page)

(continued from previous page)

```
sage: print(opt)
general options for libSingular (current value 0x06000082)
```

opt**class** sage.libs.singular.option.LibSingularOptions_abstract

Bases: object

Abstract Base Class for libSingular options.

load (value=None)

EXAMPLES:

```
sage: from sage.libs.singular.option import opt as sopt
sage: bck = sopt.save(); hex(bck[0]), bck[1], bck[2]
('0x6000082', 0, 0)
sage: sopt['redTail'] = False
sage: hex(int(sopt))
'0x4000082'
sage: sopt.load(bck)
sage: sopt['redTail']
True
```

save ()

Return a triple of integers that allow reconstruction of the options.

EXAMPLES:

```
sage: from sage.libs.singular.option import opt
sage: opt['deg_bound']
0
sage: opt['red_tail']
True
sage: s = opt.save()
sage: opt['deg_bound'] = 2
sage: opt['red_tail'] = False
sage: opt['deg_bound']
2
sage: opt['red_tail']
False
sage: opt.load(s)
sage: opt['deg_bound']
0
sage: opt['red_tail']
True
sage: opt.reset_default() # needed to avoid side effects
```

class sage.libs.singular.option.LibSingularVerboseOptionsBases: *LibSingularOptions_abstract*

Pythonic Interface to libSingular's verbosity options.

Supported options are:

- mem – shows memory usage in square brackets
- yacc – only available in debug version
- redefine – warns about variable redefinitions

- `reading` – shows the number of characters read from a file
- `loadLib` or `load_lib` – shows loading of libraries
- `debugLib` or `debug_lib` – warns about syntax errors when loading a library
- `loadProc` or `load_proc` – shows loading of procedures from libraries
- `defRes` or `def_res` – shows the names of the syzygy modules while converting resolution to list
- `usage` – shows correct usage in error messages
- `Imap` or `imap` – shows the mapping of variables with the `fetch` and `imap` commands
- `notWarnSB` or `not_warn_sb` – do not warn if a basis is not a standard basis
- `contentSB` or `content_sb` – avoids to divide by the content of a polynomial in `std` and related algorithms. Should usually not be used.
- `cancelunit` – avoids to divide polynomials by non-constant units in `std` in the local case. Should usually not be used

EXAMPLES:

```
sage: from sage.libs.singular.option import LibSingularVerboseOptions
sage: libsingular_verbose = LibSingularVerboseOptions()
sage: libsingular_verbose
verbosity options for libSingular (current value 0x00002851)
```

`reset_default()`

Return to `libSingular`'s default verbosity options.

EXAMPLES:

```
sage: from sage.libs.singular.option import opt_verb
sage: opt_verb['not_warn_sb']
False
sage: opt_verb['not_warn_sb'] = True
sage: opt_verb['not_warn_sb']
True
sage: opt_verb.reset_default()
sage: opt_verb['not_warn_sb']
False
```

8.6 Wrapper for Singular's Rings

AUTHORS:

- Martin Albrecht (2009-07): initial implementation
- Kwankyu Lee (2010-06): added matrix term order support
- Miguel Marco (2021): added transcendental extensions over \mathbb{Q}

`sage.libs.singular.ring.currRing_wrapper()`

Return a wrapper for the current ring, for use in debugging `ring_refcount_dict`.

EXAMPLES:

```
sage: from sage.libs.singular.ring import currRing_wrapper
sage: currRing_wrapper()
The ring pointer ...
```

`sage.libs.singular.ring.poison_currRing(frame, event, arg)`

Poison the `currRing` pointer.

This function sets the `currRing` to an illegal value. By setting it as the python debug hook, you can poison the `currRing` before every evaluated Python command (but not within Cython code).

INPUT:

- `frame, event, arg` – the standard arguments for the CPython debugger hook; they are not used

OUTPUT:

Returns itself, which ensures that `poison_currRing()` will stay in the debugger hook.

EXAMPLES:

```
sage: previous_trace_func = sys.gettrace() # None if no debugger running
sage: from sage.libs.singular.ring import poison_currRing
sage: sys.settrace(poison_currRing)
sage: sys.gettrace()
<built-in function poison_currRing>
sage: sys.settrace(previous_trace_func) # switch it off again
```

`sage.libs.singular.ring.print_currRing()`

Print the `currRing` pointer.

EXAMPLES:

```
sage: from sage.libs.singular.ring import print_currRing
sage: print_currRing() # random output
DEBUG: currRing == 0x7fc6fa6ec480

sage: from sage.libs.singular.ring import poison_currRing
sage: _ = poison_currRing(None, None, None)
sage: print_currRing()
DEBUG: currRing == 0x0
```

class `sage.libs.singular.ring.ring_wrapper_Py`

Bases: `object`

Python object wrapping the ring pointer.

This is useful to store ring pointers in Python containers.

You must not construct instances of this class yourself, use `wrap_ring()` instead.

EXAMPLES:

```
sage: from sage.libs.singular.ring import ring_wrapper_Py
sage: ring_wrapper_Py
<class 'sage.libs.singular.ring.ring_wrapper_Py'>
```

8.7 Singular's Groebner Strategy Objects

AUTHORS:

- Martin Albrecht (2009-07): initial implementation
- Michael Brickenstein (2009-07): initial implementation
- Hans Schoenemann (2009-07): initial implementation

class `sage.libs.singular.groebner_strategy.GroebnerStrategy`

Bases: `SageObject`

A Wrapper for Singular's Groebner Strategy Object.

This object provides functions for normal form computations and other functions for Groebner basis computation.

ALGORITHM:

Uses Singular via libSINGULAR

ideal ()

Return the ideal this strategy object is defined for.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(GF(32003))
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: strat.ideal()
Ideal (x + z, y + z) of Multivariate Polynomial Ring in x, y, z over Finite_
↪Field of size 32003
```

normal_form (p)

Compute the normal form of p with respect to the generators of this object.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: strat.normal_form(x*y) # indirect doctest
z^2
sage: strat.normal_form(x + 1)
-z + 1
```

ring ()

Return the ring this strategy object is defined over.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(GF(32003))
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: strat.ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 32003
```

class sage.libs.singular.groebner_strategy.NCGroebnerStrategy

Bases: SageObject

A Wrapper for Singular's Groebner Strategy Object.

This object provides functions for normal form computations and other functions for Groebner basis computation.

ALGORITHM:

Uses Singular via libSINGULAR

ideal()

Return the ideal this strategy object is defined for.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2-H.one()])
sage: strat = NCGroebnerStrategy(I)
sage: strat.ideal() == I
True
```

normal_form(p)

Compute the normal form of p with respect to the generators of this object.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: JL = H.ideal([x^3, y^3, z^3 - 4*z])
sage: JT = H.ideal([x^3, y^3, z^3 - 4*z], side='twosided')
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: SL = NCGroebnerStrategy(JL.std())
sage: ST = NCGroebnerStrategy(JT.std())
sage: SL.normal_form(x*y^2)
x*y^2
sage: ST.normal_form(x*y^2)
y*z
```

ring()

Return the ring this strategy object is defined over.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2-H.one()])
sage: strat = NCGroebnerStrategy(I)
sage: strat.ring() is H
True
```

sage.libs.singular.groebner_strategy.unpickle_GroebnerStrategy0(I)

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(GF(32003))
```

(continues on next page)

(continued from previous page)

```
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: loads(dumps(strat)) == strat # indirect doctest
True
```

sage.libs.singular.groebner_strategy.unpickle_NCGroebnerStrategy0(I)

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2-H.one()])
sage: strat = NCGroebnerStrategy(I)
sage: loads(dumps(strat)) == strat # indirect doctest
True
```


9.1 Context Managers for LibGAP

This module implements a context manager for global variables. This is useful since the behavior of GAP is sometimes controlled by global variables, which you might want to switch to a different value for a computation. Here is an example how you are suppose to use it from your code. First, let us set a dummy global variable for our example:

```
sage: libgap.set_global('FooBar', 123)
```

Then, if you want to switch the value momentarily you can write:

```
sage: with libgap.global_context('FooBar', 'test'):  
.....:     print(libgap.get_global('FooBar'))  
test
```

Afterward, the global variable reverts to the previous value:

```
sage: print(libgap.get_global('FooBar'))  
123
```

The value is reset even if exceptions occur:

```
sage: with libgap.global_context('FooBar', 'test'):  
.....:     print(libgap.get_global('FooBar'))  
.....:     raise ValueError(libgap.get_global('FooBar'))  
Traceback (most recent call last):  
...  
ValueError: test  
sage: print(libgap.get_global('FooBar'))  
123
```

class `sage.libs.gap.context_managers.GlobalVariableContext` (*variable, value*)

Bases: `object`

Context manager for GAP global variables.

It is recommended that you use the `sage.libs.gap.libgap.Gap.global_context()` method and not construct objects of this class manually.

INPUT:

- `variable` – string; the variable name
- `value` – anything that defines a GAP object

EXAMPLES:

```

sage: libgap.set_global('FooBar', 1)
sage: with libgap.global_context('FooBar', 2):
....:     print(libgap.get_global('FooBar'))
2
sage: libgap.get_global('FooBar')
1

```

9.2 Common global functions defined by GAP.

9.3 Long tests for GAP

These stress test the garbage collection inside GAP

```
sage.libs.gap.test_long.test_loop_1()
```

EXAMPLES:

```

sage: from sage.libs.gap.test_long import test_loop_1
sage: test_loop_1() # long time (up to 25s on sage.math, 2013)

```

```
sage.libs.gap.test_long.test_loop_2()
```

EXAMPLES:

```

sage: from sage.libs.gap.test_long import test_loop_2
sage: test_loop_2() # long time (10s on sage.math, 2013)

```

```
sage.libs.gap.test_long.test_loop_3()
```

EXAMPLES:

```

sage: from sage.libs.gap.test_long import test_loop_3
sage: test_loop_3() # long time (31s on sage.math, 2013)

```

9.4 Utility functions for GAP

exception `sage.libs.gap.util.GAPError`

Bases: `ValueError`

Exceptions raised by the GAP library

class `sage.libs.gap.util.ObjWrapper`

Bases: `object`

Wrapper for GAP master pointers.

EXAMPLES:

```

sage: from sage.libs.gap.util import ObjWrapper
sage: x = ObjWrapper()
sage: y = ObjWrapper()
sage: x == y
True

```



```
sage.libs.gap.util.get_owned_objects()
    Helper to access the refcount dictionary from Python code
```

9.5 Library Interface to GAP

This module implements a fast C library interface to GAP. To use it, you simply call `libgap` (the parent of all `GapElement` instances) and use it to convert Sage objects into GAP objects.

EXAMPLES:

```
sage: a = libgap(10)
sage: a
10
sage: type(a)
<class 'sage.libs.gap.element.GapElement_Integer'>
sage: a*a
100
sage: timeit('a*a') # random output
625 loops, best of 3: 898 ns per loop
```

Compared to the expect interface this is >1000 times faster:

```
sage: b = gap('10')
sage: timeit('b*b') # random output; long time
125 loops, best of 3: 2.05 ms per loop
```

If you want to evaluate GAP commands, use the `Gap.eval()` method:

```
sage: libgap.eval('List([1..10], i->i^2)')
[ 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 ]
```

not to be confused with the `libgap` call, which converts Sage objects to GAP objects, for example strings to strings:

```
sage: libgap('List([1..10], i->i^2)')
"List([1..10], i->i^2)"
sage: type(_)
<class 'sage.libs.gap.element.GapElement_String'>
```

You can usually use the `sage()` method to convert the resulting GAP element back to its Sage equivalent:

```
sage: a.sage()
10
sage: type(_)
<class 'sage.rings.integer.Integer'>

sage: libgap.eval('5/3 + 7*E(3)').sage() #_
↳needs sage.rings.number_field
7*zeta3 + 5/3

sage: gens_of_group = libgap.AlternatingGroup(4).GeneratorsOfGroup()
sage: generators = gens_of_group.sage()
sage: generators # a Sage list of Sage permutations!
[[2, 3, 1], [1, 3, 4, 2]]
sage: PermutationGroup(generators).cardinality() # computed in Sage
12
```

(continues on next page)

(continued from previous page)

```
sage: libgap.AlternatingGroup(4).Size()           # computed in GAP
12
```

We can also specify which group in Sage the permutations should consider themselves as elements of when converted to Sage:

```
sage: A4 = groups.permutation.Alternating(4)
sage: generators = gens_of_group.sage(parent=A4); generators
[(1,2,3), (2,3,4)]
sage: all(gen.parent() is A4 for gen in generators)
True
```

So far, the following GAP data types can be directly converted to the corresponding Sage datatype:

1. GAP booleans `true` / `false` to Sage booleans `True` / `False`. The third GAP boolean value `fail` raises a `ValueError`.
2. GAP integers to Sage integers.
3. GAP rational numbers to Sage rational numbers.
4. GAP cyclotomic numbers to Sage cyclotomic numbers.
5. GAP permutations to Sage permutations.
6. The GAP containers `List` and `rec` are converted to Sage containers `list` and `dict`. Furthermore, the `sage()` method is applied recursively to the entries.

Special support is available for the GAP container classes. GAP lists can be used as follows:

```
sage: lst = libgap([1,5,7]); lst
[ 1, 5, 7 ]
sage: type(lst)
<class 'sage.libs.gap.element.GapElement_List'>
sage: len(lst)
3
sage: lst[0]
1
sage: [ x^2 for x in lst ]
[1, 25, 49]
sage: type(_[0])
<class 'sage.libs.gap.element.GapElement_Integer'>
```

Note that you can access the elements of GAP `List` objects as you would expect from Python (with indexing starting at 0), but the elements are still of type `GapElement`. The other GAP container type are records, which are similar to Python dictionaries. You can construct them directly from Python dictionaries:

```
sage: libgap({'a':123, 'b':456})
rec( a := 123, b := 456 )
```

Or get them as results of computations:

```
sage: rec = libgap.eval('rec(a:=123, b:=456, Sym3:=SymmetricGroup(3))')
sage: rec['Sym3']
Sym( [ 1 .. 3 ] )
sage: dict(rec)
{'Sym3': Sym( [ 1 .. 3 ] ), 'a': 123, 'b': 456}
```

The output is a Sage dictionary whose keys are Sage strings and whose Values are instances of `GapElement()`. So, for example, `rec['a']` is not a Sage integer. To recursively convert the entries into Sage objects, you should use the `sage()` method:

```
sage: rec.sage()
{'Sym3': NotImplementedError('cannot construct equivalent Sage object'...),
 'a': 123,
 'b': 456}
```

Now `rec['a']` is a Sage integer. We have not implemented the conversion of the GAP symmetric group to the Sage symmetric group yet, so you end up with a `NotImplementedError` exception object. The exception is returned and not raised so that you can work with the partial result.

While we don't directly support matrices yet, you can convert them to Gap List of Lists. These lists are then easily converted into Sage using the recursive expansion of the `sage()` method:

```
sage: M = libgap.eval('BlockMatrix([[1,1,[[1, 2],[ 3, 4]]], [1,2,[[9,10],[11,12]]],_
↳[2,2,[[5, 6],[ 7, 8]]]),2,2)')
sage: M
<block matrix of dimensions (2*2)x(2*2)>
sage: M.List() # returns a GAP List of Lists
[[ 1, 2, 9, 10 ], [ 3, 4, 11, 12 ], [ 0, 0, 5, 6 ], [ 0, 0, 7, 8 ] ]
sage: M.List().sage() # returns a Sage list of lists
[[1, 2, 9, 10], [3, 4, 11, 12], [0, 0, 5, 6], [0, 0, 7, 8]]
sage: matrix(ZZ, _)
[ 1 2 9 10]
[ 3 4 11 12]
[ 0 0 5 6]
[ 0 0 7 8]
```

9.5.1 Using the GAP C library from Cython

Todo

Expand the following text

We are using the GAP API provided by the GAP project since GAP 4.10.

AUTHORS:

- William Stein, Robert Miller (2009-06-23): first version
- Volker Braun, Dmitrii Pasechnik, Ivan Andrus (2011-03-25, Sage Days 29): almost complete rewrite; first usable version.
- Volker Braun (2012-08-28, GAP/Singular workshop): update to gap-4.5.5, make it ready for public consumption.
- Dima Pasechnik (2018-09-18, GAP Days): started the port to native libgap API

```
class sage.libs.gap.libgap.Gap
```

Bases: `Parent`

The libgap interpreter object.

Note

This object must be instantiated exactly once by the `libgap`. Always use the provided `libgap` instance, and never instantiate `Gap` manually.

EXAMPLES:

```
sage: libgap.eval('SymmetricGroup(4)')
Sym( [ 1 .. 4 ] )
```

Element

alias of `GapElement`

collect ()

Manually run the garbage collector.

EXAMPLES:

```
sage: a = libgap(123)
sage: del a
sage: libgap.collect()
```

count_GAP_objects ()

Return the number of GAP objects that are being tracked by GAP.

OUTPUT: integer

EXAMPLES:

```
sage: libgap.count_GAP_objects() # random output
5
```

eval (gap_command)

Evaluate a gap command and wrap the result.

INPUT:

- `gap_command` – string containing a valid gap command without the trailing semicolon

OUTPUT: a `GapElement`

EXAMPLES:

```
sage: libgap.eval('0')
0
sage: libgap.eval('"string"')
"string"
```

function_factory (function_name)

Return a GAP function wrapper.

This is almost the same as calling `libgap.eval (function_name)`, but faster and makes it obvious in your code that you are wrapping a function.

INPUT:

- `function_name` – string; the name of a GAP function

OUTPUT:

A function wrapper *GapElement_Function* for the GAP function. Calling it from Sage is equivalent to calling the wrapped function from GAP.

EXAMPLES:

```
sage: libgap.function_factory('Print')
<Gap function "Print">
```

get_global (*variable*)

Get a GAP global variable.

INPUT:

- *variable* – string; the variable name

OUTPUT:

A *GapElement* wrapping the GAP output. A *ValueError* is raised if there is no such variable in GAP.

EXAMPLES:

```
sage: libgap.set_global('FooBar', 1)
sage: libgap.get_global('FooBar')
1
sage: libgap.unset_global('FooBar')
sage: libgap.get_global('FooBar')
NULL
```

global_context (*variable, value*)

Temporarily change a global variable.

INPUT:

- *variable* – string; the variable name
- *value* – anything that defines a GAP object

OUTPUT: a context manager that sets/reverts the given global variable

EXAMPLES:

```
sage: libgap.set_global('FooBar', 1)
sage: with libgap.global_context('FooBar', 2):
....:     print(libgap.get_global('FooBar'))
2
sage: libgap.get_global('FooBar')
1
```

load_package (*pkg*)

If loading fails, raise a *RuntimeError* exception.

one ()

Return (integer) one in GAP.

EXAMPLES:

```
sage: libgap.one()
1
sage: parent(_)
C library interface to GAP
```

set_global (*variable, value*)

Set a GAP global variable.

INPUT:

- *variable* – string; the variable name
- *value* – anything that defines a GAP object

EXAMPLES:

```
sage: libgap.set_global('FooBar', 1)
sage: libgap.get_global('FooBar')
1
sage: libgap.unset_global('FooBar')
sage: libgap.get_global('FooBar')
NULL
```

set_seed (*seed=None*)

Reseed the standard GAP pseudo-random sources with the given seed.

Uses a random seed given by `current_randstate().ZZ_seed()` if `seed=None`. Otherwise the seed should be an integer.

EXAMPLES:

```
sage: libgap.set_seed(0)
0
sage: [libgap.Random(1, 10) for i in range(5)]
[2, 3, 3, 4, 2]
```

show ()

Return statistics about the GAP owned object list.

This includes the total memory allocated by GAP as returned by `libgap.eval('TotalMemoryAllocated()')`, as well as garbage collection / object count statistics as returned by `libgap.eval('GasmanStatistics')`, and finally the total number of GAP objects held by Sage as *GapElement* instances.

The value `livekb + deadkb` will roughly equal the total memory allocated for GAP objects (see `libgap.eval('TotalMemoryAllocated()')`).

Note

Slight complication is that we want to do it without accessing libgap objects, so we don't create new *GapElements* as a side effect.

EXAMPLES:

```
sage: a = libgap(123)
sage: b = libgap(456)
sage: c = libgap(789)
sage: del b
sage: libgap.collect()
sage: libgap.show() # random output
{'gasman_stats': {'full': {'cumulative': 110,
  'deadbags': 321400,
  'deadkb': 12967,
```

(continues on next page)

(continued from previous page)

```
'freekb': 15492,
'livebags': 396645,
'livekb': 37730,
'time': 110,
'totalkb': 65536},
'nfull': 1,
'npartial': 1},
'nelements': 23123,
'total_alloc': 3234234}
```

unset_global (*variable*)

Remove a GAP global variable.

INPUT:

- *variable* – string; the variable name

EXAMPLES:

```
sage: libgap.set_global('FooBar', 1)
sage: libgap.get_global('FooBar')
1
sage: libgap.unset_global('FooBar')
sage: libgap.get_global('FooBar')
NULL
```

zero ()

Return (integer) zero in GAP.

OUTPUT: a GapElement

EXAMPLES:

```
sage: libgap.zero()
0
```

9.6 Short tests for GAP

`sage.libs.gap.test.test_write_to_file()`

Test that libgap can write to files.

See [Issue #16502](#), [Issue #15833](#).

EXAMPLES:

```
sage: from sage.libs.gap.test import test_write_to_file
sage: test_write_to_file()
```

9.7 GAP element wrapper

This document describes the individual wrappers for various GAP elements. For general information about GAP, you should read the [libgap](#) module documentation.

class `sage.libs.gap.element.GapElement`

Bases: `RingElement`

Wrapper for all Gap objects.

Note

In order to create `GapElements` you should use the `libgap` instance (the parent of all Gap elements) to convert things into `GapElement`. You must not create `GapElement` instances manually.

EXAMPLES:

```
sage: libgap(0)
0
```

If Gap finds an error while evaluating, a `GAPError` exception is raised:

```
sage: libgap.eval('1/0')
Traceback (most recent call last):
...
GAPError: Error, Rational operations: <divisor> must not be zero
```

Also, a `GAPError` is raised if the input is not a simple expression:

```
sage: libgap.eval('1; 2; 3')
Traceback (most recent call last):
...
GAPError: can only evaluate a single statement
```

deepcopy (*mut*)

Return a deepcopy of this Gap object.

Note that this is the same thing as calling `StructuralCopy` but much faster.

INPUT:

- `mut` – boolean; whether to return a mutable copy

EXAMPLES:

```
sage: a = libgap([[0,1],[2,3]])
sage: b = a.deepcopy(1)
sage: b[0,0] = 5
sage: a
[[ 0, 1 ], [ 2, 3 ]]
sage: b
[[ 5, 1 ], [ 2, 3 ]]

sage: l = libgap([0,1])
sage: l.deepcopy(0).IsMutable()
false
```

(continues on next page)

(continued from previous page)

```
sage: l.deepcopy(1).IsMutable()
true
```

is_bool()

Return whether the wrapped GAP object is a GAP boolean.

OUTPUT: boolean

EXAMPLES:

```
sage: libgap(True).is_bool()
True
```

is_function()

Return whether the wrapped GAP object is a function.

OUTPUT: boolean

EXAMPLES:

```
sage: a = libgap.eval("NormalSubgroups")
sage: a.is_function()
True
sage: a = libgap(2/3)
sage: a.is_function()
False
```

is_list()

Return whether the wrapped GAP object is a GAP List.

OUTPUT: boolean

EXAMPLES:

```
sage: libgap.eval('[1, 2, , , 5]').is_list()
True
sage: libgap.eval('3/2').is_list()
False
```

is_permutation()

Return whether the wrapped GAP object is a GAP permutation.

OUTPUT: boolean

EXAMPLES:

```
sage: perm = libgap.PermList( libgap([1,5,2,3,4]) ); perm
(2, 5, 4, 3)
sage: perm.is_permutation()
True
sage: libgap('this is a string').is_permutation()
False
```

is_record()

Return whether the wrapped GAP object is a GAP record.

OUTPUT: boolean

EXAMPLES:

```
sage: libgap.eval('[1, 2,,,, 5]').is_record()
False
sage: libgap.eval('rec(a:=1, b:=3)').is_record()
True
```

is_string()

Return whether the wrapped GAP object is a GAP string.

OUTPUT: boolean

EXAMPLES:

```
sage: libgap('this is a string').is_string()
True
```

sage()

Return the Sage equivalent of the *GapElement*.

EXAMPLES:

```
sage: libgap(1).sage()
1
sage: type(_)
<class 'sage.rings.integer.Integer'>

sage: libgap(3/7).sage()
3/7
sage: type(_)
<class 'sage.rings.rational.Rational'>

sage: libgap.eval('5 + 7*E(3)').sage()
7*zeta3 + 5

sage: libgap(Infinity).sage()
+Infinity
sage: libgap(-Infinity).sage()
-Infinity

sage: libgap(True).sage()
True
sage: libgap(False).sage()
False
sage: type(_)
<... 'bool'>

sage: libgap('this is a string').sage()
'this is a string'
sage: type(_)
<... 'str'>

sage: x = libgap.Integers.Indeterminate("x")

sage: p = x^2 - 2*x + 3
sage: p.sage()
x^2 - 2*x + 3
sage: p.sage().parent()
Univariate Polynomial Ring in x over Integer Ring
```

(continues on next page)

(continued from previous page)

```

sage: p = x^-2 + 3*x
sage: p.sage()
x^-2 + 3*x
sage: p.sage().parent()
Univariate Laurent Polynomial Ring in x over Integer Ring

sage: p = (3 * x^2 + x) / (x^2 - 2)
sage: p.sage()
(3*x^2 + x)/(x^2 - 2)
sage: p.sage().parent()
Fraction Field of Univariate Polynomial Ring in x over Integer Ring

sage: G0 = libgap.SymplecticGroup(4,2)
sage: P = G0.IsomorphismFpGroup().Range()
sage: G = P.sage()
sage: G.gap() == P
True

sage: F0 = libgap.FreeGroup(2)
sage: F = F0.sage()
sage: F.gap() is F0
True

```

class sage.libs.gap.element.GapElement_Boolean

Bases: *GapElement*

Derived class of GapElement for GAP boolean values.

EXAMPLES:

```

sage: b = libgap(True)
sage: type(b)
<class 'sage.libs.gap.element.GapElement_Boolean'>

```

sage()

Return the Sage equivalent of the *GapElement*.

OUTPUT:

A Python boolean if the values is either true or false. GAP booleans can have the third value `Fail`, in which case a `ValueError` is raised.

EXAMPLES:

```

sage: b = libgap.eval('true'); b
true
sage: type(_)
<class 'sage.libs.gap.element.GapElement_Boolean'>
sage: b.sage()
True
sage: type(_)
<... 'bool'>

sage: libgap.eval('fail')
fail
sage: _.sage()
Traceback (most recent call last):

```

(continues on next page)

(continued from previous page)

```
...
ValueError: the GAP boolean value "fail" cannot be represented in Sage
```

class sage.libs.gap.element.**GapElement_Cyclotomic**

Bases: *GapElement*

Derived class of *GapElement* for GAP universal cyclotomics.

EXAMPLES:

```
sage: libgap.eval('E(3)')
E(3)
sage: type(_)
<class 'sage.libs.gap.element.GapElement_Cyclotomic'>
```

sage (*ring=None*)

Return the Sage equivalent of the *GapElement_Cyclotomic*.

INPUT:

- *ring* – a Sage cyclotomic field or None (default); if not specified, a suitable minimal cyclotomic field will be constructed

OUTPUT: a Sage cyclotomic field element

EXAMPLES:

```
sage: n = libgap.eval('E(3)')
sage: n.sage()
zeta3
sage: parent(_)
Cyclotomic Field of order 3 and degree 2

sage: n.sage(ring=CyclotomicField(6))
zeta6 - 1

sage: libgap.E(3).sage(ring=CyclotomicField(3))
zeta3
sage: libgap.E(3).sage(ring=CyclotomicField(6))
zeta6 - 1
```

class sage.libs.gap.element.**GapElement_FiniteField**

Bases: *GapElement*

Derived class of *GapElement* for GAP finite field elements.

EXAMPLES:

```
sage: libgap.eval('Z(5)^2')
Z(5)^2
sage: type(_)
<class 'sage.libs.gap.element.GapElement_FiniteField'>
```

lift ()

Return an integer lift.

OUTPUT:

The smallest positive *GapElement_Integer* that equals *self* in the prime finite field.

EXAMPLES:

```
sage: n = libgap.eval('Z(5)^2')
sage: n.lift()
4
sage: type(_)
<class 'sage.libs.gap.element.GapElement_Integer'>

sage: n = libgap.eval('Z(25)')
sage: n.lift()
Traceback (most recent call last):
TypeError: not in prime subfield
```

sage (*ring=None, var='a'*)

Return the Sage equivalent of the *GapElement_FiniteField*.

INPUT:

- *ring* – a Sage finite field or None (default). The field to return *self* in. If not specified, a suitable finite field will be constructed.

OUTPUT:

A Sage finite field element. The isomorphism is chosen such that the *GapPrimitiveRoot()* maps to the Sage *multiplicative_generator()*.

EXAMPLES:

```
sage: n = libgap.eval('Z(25)^2')
sage: n.sage()
a + 3
sage: parent(_)
Finite Field in a of size 5^2

sage: n.sage(ring=GF(5))
Traceback (most recent call last):
...
ValueError: the given ring is incompatible ...
```

class `sage.libs.gap.element.GapElement_Float`

Bases: *GapElement*

Derived class of *GapElement* for GAP floating point numbers.

EXAMPLES:

```
sage: i = libgap(123.5)
sage: type(i)
<class 'sage.libs.gap.element.GapElement_Float'>
sage: RDF(i)
123.5
sage: float(i)
123.5
```

sage (*ring=None*)

Return the Sage equivalent of the *GapElement_Float*.

- *ring* – a floating point field or None (default); if not specified, the default Sage RDF is used

OUTPUT: a Sage double precision floating point number

EXAMPLES:

```
sage: a = libgap.eval("Float(3.25)").sage()
sage: a
3.25
sage: parent(a)
Real Double Field
```

class `sage.libs.gap.element.GapElement_Function`

Bases: *GapElement*

Derived class of *GapElement* for GAP functions.

EXAMPLES:

```
sage: f = libgap.Cycles
sage: type(f)
<class 'sage.libs.gap.element.GapElement_Function'>
```

class `sage.libs.gap.element.GapElement_Integer`

Bases: *GapElement*

Derived class of *GapElement* for GAP integers.

EXAMPLES:

```
sage: i = libgap(123)
sage: type(i)
<class 'sage.libs.gap.element.GapElement_Integer'>
sage: ZZ(i)
123
```

is_C_int()

Return whether the wrapped GAP object is a immediate GAP integer.

An immediate integer is one that is stored as a C integer, and is subject to the usual size limits. Larger integers are stored in GAP as GMP integers.

OUTPUT: boolean

EXAMPLES:

```
sage: n = libgap(1)
sage: type(n)
<class 'sage.libs.gap.element.GapElement_Integer'>
sage: n.is_C_int()
True
sage: n.IsInt()
true

sage: N = libgap(2^130)
sage: type(N)
<class 'sage.libs.gap.element.GapElement_Integer'>
sage: N.is_C_int()
False
sage: N.IsInt()
true
```

sage (*ring=None*)

Return the Sage equivalent of the *GapElement_Integer*.

- `ring` – integer ring or `None` (default); if not specified, the default Sage integer ring is used

OUTPUT: a Sage integer

EXAMPLES:

```
sage: libgap([ 1, 3, 4 ]).sage()
[1, 3, 4]
sage: all( x in ZZ for x in _ )
True

sage: libgap(132).sage(ring=IntegerModRing(13))
2
sage: parent(_)
Ring of integers modulo 13
```

class `sage.libs.gap.element.GapElement_IntegerMod`

Bases: *GapElement*

Derived class of *GapElement* for GAP integers modulo an integer.

EXAMPLES:

```
sage: n = IntegerModRing(123)(13)
sage: i = libgap(n)
sage: type(i)
<class 'sage.libs.gap.element.GapElement_IntegerMod'>
```

lift()

Return an integer lift.

OUTPUT:

A *GapElement_Integer* that equals `self` in the integer mod ring.

EXAMPLES:

```
sage: n = libgap.eval('One(ZmodnZ(123)) * 13')
sage: n.lift()
13
sage: type(_)
<class 'sage.libs.gap.element.GapElement_Integer'>
```

sage (*ring=None*)

Return the Sage equivalent of the *GapElement_IntegerMod*.

INPUT:

- `ring` – Sage integer mod ring or `None` (default); if not specified, a suitable integer mod ring is used automatically

OUTPUT: a Sage integer modulo another integer

EXAMPLES:

```
sage: n = libgap.eval('One(ZmodnZ(123)) * 13')
sage: n.sage()
13
sage: parent(_)
Ring of integers modulo 123
```

class sage.libs.gap.element.GapElement_List

Bases: *GapElement*

Derived class of GapElement for GAP Lists.

Note

Lists are indexed by $0..len(l) - 1$, as expected from Python. This differs from the GAP convention where lists start at 1.

EXAMPLES:

```
sage: lst = libgap.SymmetricGroup(3).List(); lst
[ (), (1,3), (1,2,3), (2,3), (1,3,2), (1,2) ]
sage: type(lst)
<class 'sage.libs.gap.element.GapElement_List'>
sage: len(lst)
6
sage: lst[3]
(2,3)
```

We can easily convert a Gap List object into a Python list:

```
sage: list(lst)
[ (), (1,3), (1,2,3), (2,3), (1,3,2), (1,2) ]
sage: type(_)
<... 'list'>
```

Range checking is performed:

```
sage: lst[10]
Traceback (most recent call last):
...
IndexError: index out of range.
```

matrix (*ring=None*)

Return the list as a matrix.

GAP does not have a special matrix data type, they are just lists of lists. This function converts a GAP list of lists to a Sage matrix.

OUTPUT: a Sage matrix

EXAMPLES:

```
sage: F = libgap.GF(4)
sage: a = F.PrimitiveElement()
sage: m = libgap([[a, a^0], [0*a, a^2]]); m
[ [ Z(2^2), Z(2)^0 ],
  [ 0*Z(2), Z(2^2)^2 ] ]
sage: m.IsMatrix()
true
sage: matrix(m)
[  a   1]
[  0 a + 1]
sage: matrix(GF(4, 'B'), m)
[  B   1]
```

(continues on next page)

(continued from previous page)

```

[ 0 B + 1]

sage: M = libgap.eval('SL(2,GF(5))').GeneratorsOfGroup()[1]
sage: type(M)
<class 'sage.libs.gap.element.GapElement_List'>
sage: M[0][0]
Z(5)^2
sage: M.IsMatrix()
true
sage: M.matrix()
[4 1]
[4 0]

```

sage (**kws)Return the Sage equivalent of the *GapElement*.

OUTPUT: a Python list

EXAMPLES:

```

sage: libgap([ 1, 3, 4 ]).sage()
[1, 3, 4]
sage: all( x in ZZ for x in _ )
True

```

vector (ring=None)

Return the list as a vector.

GAP does not have a special vector data type, they are just lists. This function converts a GAP list to a Sage vector.

OUTPUT: a Sage vector

EXAMPLES:

```

sage: F = libgap.GF(4)
sage: a = F.PrimitiveElement()
sage: m = libgap([0*a, a, a^3, a^2]); m
[ 0*Z(2), Z(2^2), Z(2)^0, Z(2^2)^2 ]
sage: type(m)
<class 'sage.libs.gap.element.GapElement_List'>
sage: m[3]
Z(2^2)^2
sage: vector(m)
(0, a, 1, a + 1)
sage: vector(GF(4, 'B'), m)
(0, B, 1, B + 1)

```

class sage.libs.gap.element.**GapElement_MethodProxy**Bases: *GapElement_Function*Helper class returned by *GapElement.__getattr__*.Derived class of *GapElement* for GAP functions. Like its parent, you can call instances to implement function call syntax. The only difference is that a fixed first argument is prepended to the argument list.

EXAMPLES:

```

sage: lst = libgap([])
sage: lst.Add
<Gap function "Add">
sage: type(_)
<class 'sage.libs.gap.element.GapElement_MethodProxy'>
sage: lst.Add(1)
sage: lst
[ 1 ]

```

class sage.libs.gap.element.**GapElement_Permutation**

Bases: *GapElement*

Derived class of *GapElement* for GAP permutations.

Note

Permutations in GAP act on the numbers starting with 1.

EXAMPLES:

```

sage: perm = libgap.eval('(1,5,2)(4,3,8)')
sage: type(perm)
<class 'sage.libs.gap.element.GapElement_Permutation'>

```

sage (*parent=None*)

Return the Sage equivalent of the *GapElement*.

If the permutation group is given as parent, this method is *much* faster.

EXAMPLES:

```

sage: perm_gap = libgap.eval('(1,5,2)(4,3,8)'); perm_gap
(1,5,2)(3,8,4)
sage: perm_gap.sage()
[5, 1, 8, 3, 2, 6, 7, 4]
sage: type(_)
<class 'sage.combinat.permutation.StandardPermutations_all_with_category.
↳element_class'>
sage: perm_gap.sage(PermutationGroup([(1,2),(1,2,3,4,5,6,7,8)]))
(1,5,2)(3,8,4)
sage: type(_)
<class 'sage.groups.perm_gps.permgroup_element.PermutationGroupElement'>

```

class sage.libs.gap.element.**GapElement_Rational**

Bases: *GapElement*

Derived class of *GapElement* for GAP rational numbers.

EXAMPLES:

```

sage: r = libgap(123/456)
sage: type(r)
<class 'sage.libs.gap.element.GapElement_Rational'>

```

sage (*ring=None*)

Return the Sage equivalent of the *GapElement*.

INPUT:

- `ring` – the Sage rational ring or `None` (default); if not specified, the rational ring is used automatically

OUTPUT: a Sage rational number

EXAMPLES:

```
sage: r = libgap(123/456); r
41/152
sage: type(_)
<class 'sage.libs.gap.element.GapElement_Rational'>
sage: r.sage()
41/152
sage: type(_)
<class 'sage.rings.rational.Rational'>
```

class `sage.libs.gap.element.GapElement_Record`

Bases: `GapElement`

Derived class of `GapElement` for GAP records.

EXAMPLES:

```
sage: rec = libgap.eval('rec(a:=123, b:=456)')
sage: type(rec)
<class 'sage.libs.gap.element.GapElement_Record'>
sage: len(rec)
2
sage: rec['a']
123
```

We can easily convert a Gap `rec` object into a Python dict:

```
sage: dict(rec)
{'a': 123, 'b': 456}
sage: type(_)
<... 'dict'>
```

Range checking is performed:

```
sage: rec['no_such_element']
Traceback (most recent call last):
...
GAPError: Error, Record Element: '<rec>.no_such_element' must have an assigned_
↪value
```

record_name_to_index (*name*)

Convert string to GAP record index.

INPUT:

- `py_name` – a python string

OUTPUT:

A `UInt`, which is a GAP hash of the string. If this is the first time the string is encountered, a new integer is returned(!)

EXAMPLES:

```
sage: rec = libgap.eval('rec(first:=123, second:=456)')
sage: rec.record_name_to_index('first') # random output
1812
sage: rec.record_name_to_index('no_such_name') # random output
3776
```

sage()

Return the Sage equivalent of the *GapElement*.

EXAMPLES:

```
sage: libgap.eval('rec(a:=1, b:=2)').sage()
{'a': 1, 'b': 2}
sage: all( isinstance(key, str) and val in ZZ for key, val in _.items() )
True

sage: rec = libgap.eval('rec(a:=123, b:=456, Sym3:=SymmetricGroup(3))')
sage: rec.sage()
{'Sym3': NotImplementedError('cannot construct equivalent Sage object...'),
 'a': 123,
 'b': 456}
```

class sage.libs.gap.element.GapElement_RecordIterator

Bases: object

Iterator for *GapElement_Record*.

Since Cython does not support generators yet, we implement the older iterator specification with this auxiliary class.

INPUT:

- `rec` – the *GapElement_Record* to iterate over

EXAMPLES:

```
sage: rec = libgap.eval('rec(a:=123, b:=456)')
sage: sorted(rec)
[('a', 123), ('b', 456)]
sage: dict(rec)
{'a': 123, 'b': 456}
```

class sage.libs.gap.element.GapElement_Ring

Bases: *GapElement*

Derived class of *GapElement* for GAP rings (parents of ring elements).

EXAMPLES:

```
sage: i = libgap(ZZ)
sage: type(i)
<class 'sage.libs.gap.element.GapElement_Ring'>
```

ring_cyclotomic()

Construct an integer ring.

EXAMPLES:

```
sage: libgap.CyclotomicField(6).ring_cyclotomic()
Cyclotomic Field of order 3 and degree 2
```

ring_finite_field(*var='a'*)

Construct an integer ring.

EXAMPLES:

```
sage: libgap.GF(3,2).ring_finite_field(var='A')
Finite Field in A of size 3^2
```

ring_integer()

Construct the Sage integers.

EXAMPLES:

```
sage: libgap.eval('Integers').ring_integer()
Integer Ring
```

ring_integer_mod()

Construct a Sage integer mod ring.

EXAMPLES:

```
sage: libgap.eval('ZmodnZ(15)').ring_integer_mod()
Ring of integers modulo 15
```

ring_polynomial()

Construct a polynomial ring.

EXAMPLES:

```
sage: B = libgap(QQ['x'])
sage: B.ring_polynomial()
Univariate Polynomial Ring in x over Rational Field

sage: B = libgap(ZZ['x','y'])
sage: B.ring_polynomial()
Multivariate Polynomial Ring in x, y over Integer Ring
```

ring_rational()

Construct the Sage rationals.

EXAMPLES:

```
sage: libgap.eval('Rationals').ring_rational()
Rational Field
```

sage(***kws*)

Return the Sage equivalent of the *GapElement_Ring*.

INPUT:

- ***kws* – keywords that are passed on to the `ring_` method

OUTPUT: a Sage ring

EXAMPLES:

```
sage: libgap.eval('Integers').sage()
Integer Ring
```

(continues on next page)

(continued from previous page)

```

sage: libgap.eval('Rationals').sage()
Rational Field

sage: libgap.eval('ZmodnZ(15)').sage()
Ring of integers modulo 15

sage: libgap.GF(3,2).sage(var='A')
Finite Field in A of size 3^2

sage: libgap.CyclotomicField(6).sage()
Cyclotomic Field of order 3 and degree 2

sage: libgap(QQ['x','y']).sage()
Multivariate Polynomial Ring in x, y over Rational Field

```

class sage.libs.gap.element.GapElement_String

Bases: *GapElement*

Derived class of GapElement for GAP strings.

EXAMPLES:

```

sage: s = libgap('string')
sage: type(s)
<class 'sage.libs.gap.element.GapElement_String'>
sage: s
"string"
sage: print(s)
string

```

sage()

Convert this *GapElement_String* to a Python string.

OUTPUT: a Python string

EXAMPLES:

```

sage: s = libgap.eval(' "string" '); s
"string"
sage: type(_)
<class 'sage.libs.gap.element.GapElement_String'>
sage: str(s)
'string'
sage: s.sage()
'string'
sage: type(_)
<class 'str'>

```

9.8 LibGAP Workspace Support

The single purpose of this module is to provide the location of the libgap saved workspace and a time stamp to invalidate saved workspaces.

`sage.libs.gap.saved_workspace.timestamp()`

Return a time stamp for (lib)gap.

OUTPUT:

Float. Unix timestamp of the most recently changed GAP/LibGAP file(s). In particular, the timestamp increases whenever a gap package is added.

EXAMPLES:

```
sage: from sage.libs.gap.saved_workspace import timestamp
sage: timestamp()      # random output
1406642467.25684
sage: type(timestamp())
<... 'float'>
```

`sage.libs.gap.saved_workspace.workspace(name='workspace')`

Return the filename of the gap workspace and whether it is up to date.

INPUT:

- name – string; a name that will become part of the workspace filename

OUTPUT:

Pair consisting of a string and a boolean. The string is the filename of the saved libgap workspace (or that it should have if it doesn't exist). The boolean is whether the workspace is up-to-date. You may use the workspace file only if the boolean is True.

EXAMPLES:

```
sage: from sage.libs.gap.saved_workspace import workspace
sage: ws, up_to_date = workspace()
sage: ws
'../../gap/libgap-workspace-...'
sage: isinstance(up_to_date, bool)
True
```


10.1 Interface between flint matrices and linbox

This module only contains C++ code (and the interface is fully C compatible). It basically contains what used to be in the LinBox source code under `interfaces/sage/linbox-sage.C` written by M. Albrecht and C. Pernet. The functions available are:

- `void linbox_fmpz_mat_mul(fmpz_mat_t C, fmpz_mat_t A, fmpz_mat_t B)`: set `C` to be the result of the multiplication $A * B$
- `void linbox_fmpz_mat_charpoly(fmpz_poly_t cp, fmpz_mat_t A)`: set `cp` to be the characteristic polynomial of the square matrix `A`
- `void linbox_fmpz_mat_minpoly(fmpz_poly_t mp, fmpz_mat_t A)`: set `mp` to be the minimal polynomial of the square matrix `A`
- `size_t linbox_fmpz_mat_rank(fmpz_mat_t A)`: return the rank of the matrix `A`
- `void linbox_fmpz_mat_det(fmpz_t det, fmpz_mat_t A)`: set `det` to the determinant of the square matrix `A`

11.1 An interface to Anders Buch's Littlewood-Richardson Calculator

`lrcalc`

The “Littlewood-Richardson Calculator” is a C library for fast computation of Littlewood-Richardson (LR) coefficients and products of Schubert polynomials. It handles single LR coefficients, products of and coproducts of Schur functions, skew Schur functions, and fusion products. All of the above are achieved by counting LR (skew)-tableaux (also called Yamanouchi (skew)-tableaux) of appropriate shape and content by iterating through them. Additionally, `lrcalc` handles products of Schubert polynomials.

The web page of `lrcalc` is <http://sites.math.rutgers.edu/~asbuch/lrcalc/>.

The following describes the Sage interface to this library.

EXAMPLES:

```
sage: import sage.libs.lrcalc.lrcalc as lrcalc
```

Compute a single Littlewood-Richardson coefficient:

```
sage: lrcalc.lrccoef([3,2,1],[2,1],[2,1])
2
```

Compute a product of Schur functions; return the coefficients in the Schur expansion:

```
sage: lrcalc.mult([2,1],[2,1])
{[2, 2, 1, 1]: 1,
 [2, 2, 2]: 1,
 [3, 1, 1, 1]: 1,
 [3, 2, 1]: 2,
 [3, 3]: 1,
 [4, 1, 1]: 1,
 [4, 2]: 1}
```

Same product, but include only partitions with at most 3 rows. This corresponds to computing in the representation ring of $\mathfrak{gl}(3)$:

```
sage: lrcalc.mult([2,1],[2,1],3)
{[2, 2, 2]: 1, [3, 2, 1]: 2, [3, 3]: 1, [4, 1, 1]: 1, [4, 2]: 1}
```

We can also compute the fusion product, here for $\mathfrak{sl}(3)$ and level 2:

```
sage: lrcalc.mult([3,2,1],[3,2,1],3,2)
{[4, 4, 4]: 1, [5, 4, 3]: 1}
```

Compute the expansion of a skew Schur function:

```
sage: lrcalc.skew([3,2,1],[2,1])
{[1, 1, 1]: 1, [2, 1]: 2, [3]: 1}
```

Compute the coproduct of a Schur function:

```
sage: lrcalc.coproduct([3,2,1])
{([1, 1, 1], [2, 1]): 1,
 ([2, 1], [2, 1]): 2,
 ([2, 1], [3]): 1,
 ([2, 1, 1], [1, 1]): 1,
 ([2, 1, 1], [2]): 1,
 ([2, 2], [1, 1]): 1,
 ([2, 2], [2]): 1,
 ([2, 2, 1], [1]): 1,
 ([3, 1], [1, 1]): 1,
 ([3, 1], [2]): 1,
 ([3, 1, 1], [1]): 1,
 ([3, 2], [1]): 1,
 ([3, 2, 1], [1]): 1}
```

Multiply two Schubert polynomials:

```
sage: lrcalc.mult_schubert([4,2,1,3],[1,4,2,5,3])
{[4, 5, 1, 3, 2]: 1,
 [5, 3, 1, 4, 2]: 1,
 [5, 4, 1, 2, 3]: 1,
 [6, 2, 1, 4, 3, 5]: 1}
```

Same product, but include only permutations of 5 elements in the result. This corresponds to computing in the cohomology ring of $Fl(5)$:

```
sage: lrcalc.mult_schubert([4,2,1,3],[1,4,2,5,3],5)
{[4, 5, 1, 3, 2]: 1, [5, 3, 1, 4, 2]: 1, [5, 4, 1, 2, 3]: 1}
```

List all Littlewood-Richardson tableaux of skew shape μ/ν ; in this example $\mu = [3, 2, 1]$ and $\nu = [2, 1]$. Specifying a third entry $M' = \text{`maxrows`}$ restricts the alphabet to $\{1, 2, \dots, M\}$:

```
sage: list(lrcalc.lrskew([3,2,1],[2,1]))
[[[None, None, 1], [None, 1], [1]], [[None, None, 1], [None, 1], [2]],
 [[None, None, 1], [None, 2], [1]], [[None, None, 1], [None, 2], [3]]]

sage: list(lrcalc.lrskew([3,2,1],[2,1],maxrows=2))
[[[None, None, 1], [None, 1], [1]], [[None, None, 1], [None, 1], [2]],
 [[None, None, 1], [None, 2], [1]]]
```

Todo

Use this library in the `SymmetricFunctions` code, to make it easy to apply it to linear combinations of Schur functions.

See also

- `lrcoef()`

- `mult()`
- `coprod()`
- `skew()`
- `lrskew()`
- `mult_schubert()`

Underlying algorithmic in `lrcalc`

Here is some additional information regarding the main low-level C-functions in `lrcalc`. Given two partitions `outer` and `inner` with `inner` contained in `outer`, the function:

```
skewtab *st_new(vector *outer, vector *inner, vector *conts, int maxrows)
```

constructs and returns the (lexicographically) first LR skew tableau of shape `outer / inner`. Further restrictions can be imposed using `conts` and `maxrows`.

Namely, the integer `maxrows` is a bound on the integers that can be put in the tableau. The name is chosen because this will limit the partitions in the output of `skew()` or `mult()` to partitions with at most this number of rows.

The vector `conts` is the content of an empty tableau(!). More precisely, this vector is added to the usual content of a tableau whenever the content is needed. This affects which tableaux are considered LR tableaux (see `mult()` below). `conts` may also be the NULL pointer, in which case nothing is added.

The other function:

```
int *st_next(skewtab *st)
```

computes in place the (lexicographically) next skew tableau with the same constraints, or returns 0 if `st` is the last one.

For a first example, see the `skew()` function code in the `lrcalc` source code. We want to compute a skew Schur function, so create a skew LR tableau of the appropriate shape with `st_new` (with `conts = NULL`), then iterate through all the LR tableaux with `st_next()`. For each skew tableau, we use that `st->conts` is the content of the skew tableau, find this shape in the `res` hash table and add one to the value.

For a second example, see `mult(vector *sh1, vector *sh2, maxrows)`. Here we call `st_new()` with the shape `sh1 / (0)` and use `sh2` as the `conts` argument. The effect of using `sh2` in this way is that `st_next` will iterate through semistandard tableaux T of shape `sh1` such that the following tableau:

```
111111
22222  <--- minimal tableau of shape sh2
333
****
**T**
****
**
```

is a LR skew tableau, and `st->conts` contains the content of the combined tableaux.

More generally, `st_new(outer, inner, conts, maxrows)` and `st_next` can be used to compute the Schur expansion of the product $S_{\{outer/inner\}} * S_{conts}$, restricted to partitions with at most `maxrows` rows.

AUTHORS:

- Mike Hansen (2010): core of the interface

- Anne Schilling, Nicolas M. Thiéry, and Anders Buch (2011): fusion product, iterating through LR tableaux, finalization, documentation

`sage.libs.lrcalc.lrcalc.coprod` (*part*, *all=0*)

Compute the coproduct of a Schur function.

Return a linear combination of pairs of partitions representing the coproduct of the Schur function given by the partition *part*.

INPUT:

- *part* – a partition
- *all* – integer

If *all* is nonzero then all terms are included in the result. If *all* is zero, then only pairs of partitions (*part1*, *part2*) for which the weight of *part1* is greater than or equal to the weight of *part2* are included; the rest of the coefficients are redundant because Littlewood-Richardson coefficients are symmetric.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import coprod
sage: sorted(coprod([2,1]).items())
[[([1, 1], [1]), 1], ([2], [1]), 1], ([2, 1], []), 1]
```

`sage.libs.lrcalc.lrcalc.lrcoef` (*outer*, *inner1*, *inner2*)

Compute a single Littlewood-Richardson coefficient.

Return the coefficient of *outer* in the product of the Schur functions indexed by *inner1* and *inner2*.

INPUT:

- *outer* – a partition (weakly decreasing list of nonnegative integers)
- *inner1* – a partition
- *inner2* – a partition

Note

This function converts its inputs into `Partition()`'s. If you don't need these checks and your inputs are valid, then you can use `lrcoef_unsafe()`.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import lrcoef
sage: lrcoef([3,2,1], [2,1], [2,1])
2
sage: lrcoef([3,3], [2,1], [2,1])
1
sage: lrcoef([2,1,1,1,1], [2,1], [2,1])
0
```

`sage.libs.lrcalc.lrcalc.lrcoef_unsafe` (*outer*, *inner1*, *inner2*)

Compute a single Littlewood-Richardson coefficient.

Return the coefficient of *outer* in the product of the Schur functions indexed by *inner1* and *inner2*.

INPUT:

- *outer* – a partition (weakly decreasing list of nonnegative integers)

- inner1 – a partition
- inner2 – a partition

Warning

This function does not do any check on its input. If you want to use a safer version, use `lrcoef()`.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import lrcoef_unsafe
sage: lrcoef_unsafe([3,2,1], [2,1], [2,1])
2
sage: lrcoef_unsafe([3,3], [2,1], [2,1])
1
sage: lrcoef_unsafe([2,1,1,1,1], [2,1], [2,1])
0
```

`sage.libs.lrcalc.lrcalc.lrskew` (*outer*, *inner*, *weight=None*, *maxrows=-1*)

Iterate over the skew LR tableaux of shape `outer / inner`.

INPUT:

- outer – a partition
- inner – a partition
- weight – a partition (optional)
- maxrows – positive integer (optional)

OUTPUT: an iterator of `SkewTableau`

Specifying `maxrows = M` restricts the alphabet to $\{1, 2, \dots, M\}$.

Specifying `weight` returns only those tableaux of given content/weight.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import lrskew
sage: for st in lrskew([3,2,1], [2]):
.....:     st.pp()
. . 1
1 1
2
. . 1
1 2
2
. . 1
1 2
3

sage: for st in lrskew([3,2,1], [2], maxrows=2):
.....:     st.pp()
. . 1
1 1
2
. . 1
1 2
2
```

(continues on next page)

(continued from previous page)

```
sage: list(lrskew([3,2,1],[2], weight=[3,1]))
[[[None, None, 1], [1, 1], [2]]]
```

`sage.libs.lrcalc.lrcalc.mult` (*part1*, *part2*, *maxrows=None*, *level=None*, *quantum=None*)

Compute a product of two Schur functions.

Return the product of the Schur functions indexed by the partitions *part1* and *part2*.

INPUT:

- *part1* – a partition
- *part2* – a partition
- *maxrows* – integer (optional)
- *level* – integer (optional)
- *quantum* – an element of a ring (optional)

If *maxrows* is specified, then only partitions with at most this number of rows are included in the result.

If both *maxrows* and *level* are specified, then the function calculates the fusion product for $\mathfrak{sl}(\text{maxrows})$ of the given level.

If *quantum* is set, then this returns the product in the quantum cohomology ring of the Grassmannian. In particular, both *maxrows* and *level* need to be specified.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import mult
sage: mult([2],[1])
{[2]: 1}
sage: sorted(mult([2],[2]).items())
[[[2, 2], 1], ([3, 1], 1), ([4], 1)]
sage: sorted(mult([2,1],[2,1]).items())
[[[2, 2, 1, 1], 1], ([2, 2, 2], 1), ([3, 1, 1, 1], 1),
 [3, 2, 1], 2], ([3, 3], 1), ([4, 1, 1], 1), ([4, 2], 1)]
sage: sorted(mult([2,1],[2,1],maxrows=2).items())
[[[3, 3], 1], ([4, 2], 1)]
sage: mult([2,1],[3,2,1],3)
{[3, 3, 3]: 1, [4, 3, 2]: 2, [4, 4, 1]: 1, [5, 2, 2]: 1, [5, 3, 1]: 1}
sage: mult([2,1],[2,1],3,3)
{[2, 2, 2]: 1, [3, 2, 1]: 2, [3, 3]: 1, [4, 1, 1]: 1}
sage: mult([2,1],[2,1],None,3)
Traceback (most recent call last):
...
ValueError: maxrows needs to be specified if you specify the level
```

The quantum product::

```
sage: q = polygen(QQ, 'q')
sage: sorted(mult([1],[2,1], 2, 2, quantum=q).items())
[[[], q], ([2, 2], 1)]
sage: sorted(mult([2,1],[2,1], 2, 2, quantum=q).items())
[[[1, 1], q], ([2], q)]

sage: mult([2,1],[2,1], quantum=q)
Traceback (most recent call last):
```

(continues on next page)

(continued from previous page)

```
...
ValueError: missing parameters maxrows or level
```

`sage.libs.lrcalc.lrcalc.mult_schubert(w1, w2, rank=0)`

Compute a product of two Schubert polynomials.

Return a linear combination of permutations representing the product of the Schubert polynomials indexed by the permutations `w1` and `w2`.

INPUT:

- `w1` – a permutation
- `w2` – a permutation
- `rank` – integer

If `rank` is nonzero, then only permutations from the symmetric group $S(\text{rank})$ are included in the result.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import mult_schubert
sage: result = mult_schubert([3, 1, 5, 2, 4], [3, 5, 2, 1, 4])
sage: sorted(result.items())
[[([5, 4, 6, 1, 2, 3], 1), ([5, 6, 3, 1, 2, 4], 1),
 ([5, 7, 2, 1, 3, 4, 6], 1), ([6, 3, 5, 1, 2, 4], 1),
 ([6, 4, 3, 1, 2, 5], 1), ([6, 5, 2, 1, 3, 4], 1),
 ([7, 3, 4, 1, 2, 5, 6], 1), ([7, 4, 2, 1, 3, 5, 6], 1)]
```

`sage.libs.lrcalc.lrcalc.skew(outer, inner, maxrows=-1)`

Compute the Schur expansion of a skew Schur function.

Return a linear combination of partitions representing the Schur function of the skew Young diagram `outer / inner`, consisting of boxes in the partition `outer` that are not in `inner`.

INPUT:

- `outer` – a partition
- `inner` – a partition
- `maxrows` – integer or `None`

If `maxrows` is specified, then only partitions with at most this number of rows are included in the result.

EXAMPLES:

```
sage: from sage.libs.lrcalc.lrcalc import skew
sage: sorted(skew([2,1],[1]).items())
[[[1, 1], 1], ([2], 1)]
```


12.1 Utilities for Sage-mpmath interaction

Also patches some mpmath functions for speed

`sage.libs.mpmath.utils.bitcount` (*n*)
Bitcount of a Sage Integer or Python int/long.

EXAMPLES:

```
sage: from mpmath.libmp import bitcount
sage: bitcount(0)
0
sage: bitcount(1)
1
sage: bitcount(100)
7
sage: bitcount(-100)
7
sage: bitcount(2r)
2
sage: bitcount(2L)
2
```

`sage.libs.mpmath.utils.call` (*func*, **args*, ***kwargs*)

Call an mpmath function with Sage objects as inputs and convert the result back to a Sage real or complex number.

By default, a RealNumber or ComplexNumber with the current working precision of mpmath (`mpmath.mp.prec`) will be returned.

If `prec=n` is passed among the keyword arguments, the temporary working precision will be set to *n* and the result will also have this precision.

If `parent=P` is passed, `P.prec()` will be used as working precision and the result will be coerced to *P* (or the corresponding complex field if necessary).

Arguments should be Sage objects that can be coerced into RealField or ComplexField elements. Arguments may also be tuples, lists or dicts (which are converted recursively), or any type that mpmath understands natively (e.g. Python floats, strings for options).

EXAMPLES:

```
sage: import sage.libs.mpmath.all as a
sage: a.mp.prec = 53
sage: a.call(a.erf, 3+4*I)
```

(continues on next page)

(continued from previous page)

```

-120.186991395079 - 27.7503372936239*I
sage: a.call(a.polylog, 2, 1/3+4/5*I)
0.153548951541433 + 0.875114412499637*I
sage: a.call(a.barnesg, 3+4*I)
-0.000676375932234244 - 0.0000442236140124728*I
sage: a.call(a.barnesg, -4)
0.0000000000000000
sage: a.call(a.hyper, [2,3], [4,5], 1/3)
1.10703578162508
sage: a.call(a.hyper, [2,3], [4,(2,3)], 1/3)
1.95762943509305
sage: a.call(a.quad, a.erf, [0,1])
0.486064958112256
sage: a.call(a.gammainc, 3+4*I, 2/3, 1-pi*I, prec=100)
-274.18871130777160922270612331 + 101.59521032382593402947725236*I
sage: x = (3+4*I).n(100)
sage: y = (2/3).n(100)
sage: z = (1-pi*I).n(100)
sage: a.call(a.gammainc, x, y, z, prec=100)
-274.18871130777160922270612331 + 101.59521032382593402947725236*I
sage: a.call(a.erf, infinity)
1.0000000000000000
sage: a.call(a.erf, -infinity)
-1.0000000000000000
sage: a.call(a.gamma, infinity)
+infinity
sage: a.call(a.polylog, 2, 1/2, parent=RR)
0.582240526465012
sage: a.call(a.polylog, 2, 2, parent=RR)
2.46740110027234 - 2.17758609030360*I
sage: a.call(a.polylog, 2, 1/2, parent=RealField(100))
0.58224052646501250590265632016
sage: a.call(a.polylog, 2, 2, parent=RealField(100))
2.4674011002723396547086227500 - 2.1775860903036021305006888982*I
sage: a.call(a.polylog, 2, 1/2, parent=CC)
0.582240526465012
sage: type(_)
<class 'sage.rings.complex_mpf.ComplexNumber'>
sage: a.call(a.polylog, 2, 1/2, parent=RDF)
0.5822405264650125
sage: type(_)
<class 'sage.rings.real_double...RealDoubleElement...'>

```

Check that [Issue #11885](#) is fixed:

```
sage: a.call(a.ei, 1.0r, parent=float)
1.8951178163559366
```

Check that [Issue #14984](#) is fixed:

```
sage: a.call(a.log, -1.0r, parent=float)
3.141592653589793j
```

`sage.libs.mpmath.utils.from_man_exp(man, exp, prec=0, rnd='d')`

Create normalized mpf value tuple from mantissa and exponent.

With `prec > 0`, rounds the result in the desired direction if necessary.

EXAMPLES:

```
sage: from mpmath.libmp import from_man_exp
sage: from_man_exp(-6, -1)
(1, 3, 0, 2)
sage: from_man_exp(-6, -1, 1, 'd')
(1, 1, 1, 1)
sage: from_man_exp(-6, -1, 1, 'u')
(1, 1, 2, 1)
```

`sage.libs.mpmath.utils.isqrt(n)`

Square root (rounded to floor) of a Sage Integer or Python int/long. The result is a Sage Integer.

EXAMPLES:

```
sage: from mpmath.libmp import isqrt
sage: isqrt(0)
0
sage: isqrt(100)
10
sage: isqrt(10)
3
sage: isqrt(10r)
3
sage: isqrt(10L)
3
```

`sage.libs.mpmath.utils.mpmath_to_sage(x, prec)`

Convert any mpmath number (mpf or mpc) to a Sage RealNumber or ComplexNumber of the given precision.

EXAMPLES:

```
sage: import sage.libs.mpmath.all as a
sage: a.mpmath_to_sage(a.mpf('2.5'), 53)
2.500000000000000
sage: a.mpmath_to_sage(a.mpc('2.5', '-3.5'), 53)
2.500000000000000 - 3.500000000000000*I
sage: a.mpmath_to_sage(a.mpf('inf'), 53)
+infinity
sage: a.mpmath_to_sage(a.mpf('-inf'), 53)
-infinity
sage: a.mpmath_to_sage(a.mpf('nan'), 53)
NaN
sage: a.mpmath_to_sage(a.mpf('0'), 53)
0.000000000000000
```

A real example:

```
sage: RealField(100)(pi)
3.1415926535897932384626433833
sage: t = RealField(100)(pi)._mpmath(); t
mpf('3.1415926535897932')
sage: a.mpmath_to_sage(t, 100)
3.1415926535897932384626433833
```

We can ask for more precision, but the result is undefined:

```
sage: a.mpmath_to_sage(t, 140) # random
3.1415926535897932384626433832793333156440
sage: ComplexField(140)(pi)
3.1415926535897932384626433832795028841972
```

A complex example:

```
sage: ComplexField(100)([0, pi])
3.1415926535897932384626433833*I
sage: t = ComplexField(100)([0, pi])._mpmath(); t
mpc(real='0.0', imag='3.1415926535897932')
sage: sage.libs.mpmath.all.mpmath_to_sage(t, 100)
3.1415926535897932384626433833*I
```

Again, we can ask for more precision, but the result is undefined:

```
sage: sage.libs.mpmath.all.mpmath_to_sage(t, 140) # random
3.1415926535897932384626433832793333156440*I
sage: ComplexField(140)([0, pi])
3.1415926535897932384626433832795028841972*I
```

`sage.libs.mpmath.utils.normalize` (*sign, man, exp, bc, prec, rnd*)

Create normalized mpf value tuple from full list of components.

EXAMPLES:

```
sage: from mpmath.libmp import normalize
sage: normalize(0, 4, 5, 3, 53, 'n')
(0, 1, 7, 1)
```

`sage.libs.mpmath.utils.sage_to_mpmath` (*x, prec*)

Convert any Sage number that can be coerced into a RealNumber or ComplexNumber of the given precision into an mpmath mpf or mpc. Integers are currently converted to int.

Lists, tuples and dicts passed as input are converted recursively.

EXAMPLES:

```
sage: import sage.libs.mpmath.all as a
sage: a.mp.dps = 15
sage: print(a.sage_to_mpmath(2/3, 53))
0.6666666666666667
sage: print(a.sage_to_mpmath(2./3, 53))
0.6666666666666667
sage: print(a.sage_to_mpmath(3+4*I, 53))
(3.0 + 4.0j)
sage: print(a.sage_to_mpmath(1+pi, 53))
4.14159265358979
sage: a.sage_to_mpmath(infinity, 53)
mpf('+inf')
sage: a.sage_to_mpmath(-infinity, 53)
mpf('-inf')
sage: a.sage_to_mpmath(NaN, 53)
mpf('nan')
sage: a.sage_to_mpmath(0, 53)
0
sage: a.sage_to_mpmath([0.5, 1.5], 53)
[mpf('0.5'), mpf('1.5')]
```

(continues on next page)

(continued from previous page)

```
sage: a.sage_to_mpmath((0.5, 1.5), 53)
(mpf('0.5'), mpf('1.5'))
sage: a.sage_to_mpmath({'n':0.5}, 53)
{'n': mpf('0.5')}
```


13.1 Victor Shoup's NTL C++ Library

Sage provides an interface to Victor Shoup's C++ library NTL. Features of this library include *incredibly fast* arithmetic with polynomials and asymptotically fast factorization of polynomials.

14.1 Interface between Sage and PARI

14.1.1 Guide to real precision in the PARI interface

In the PARI interface, “real precision” refers to the precision of real numbers, so it is the floating-point precision. This is a non-trivial issue, since there are various interfaces for different things.

Internal representation and conversion between Sage and PARI

Real numbers in PARI have a precision associated to them, which is always a multiple of the CPU wordsize. So, it is a multiple of 32 or 64 bits. When converting from Sage to PARI, the precision is rounded up to the nearest multiple of the wordsize:

```
sage: x = 1.0
sage: x.precision()
53
sage: pari(x)
1.0000000000000000
sage: pari(x).bitprecision()
64
```

With a higher precision:

```
sage: x = RealField(100).pi()
sage: x.precision()
100
sage: pari(x).bitprecision()
128
```

When converting back to Sage, the precision from PARI is taken:

```
sage: x = RealField(100).pi()
sage: y = pari(x).sage()
sage: y
3.1415926535897932384626433832793333156
sage: parent(y)
Real Field with 128 bits of precision
```

So `pari(x).sage()` is definitely not equal to `x` since it has 28 bogus bits.

Therefore, some care must be taken when juggling reals back and forth between Sage and PARI. The correct way of avoiding this is to convert `pari(x).sage()` back into a domain with the right precision. This has to be done by the

user (or by Sage functions that use PARI library functions). For instance, if we want to use the PARI library to compute `sqrt(pi)` with a precision of 100 bits:

```
sage: # needs sage.symbolic
sage: R = RealField(100)
sage: s = R(pi); s
3.1415926535897932384626433833
sage: p = pari(s).sqrt()
sage: x = p.sage(); x # wow, more digits than I expected!
1.7724538509055160272981674833410973484
sage: x.prec() # has precision 'improved' from 100 to 128?
128
sage: x == RealField(128)(pi).sqrt() # sadly, no!
False
sage: R(x) # x should be brought back to precision 100
1.7724538509055160272981674833
sage: R(x) == s.sqrt()
True
```

Output precision for printing

Even though PARI reals have a precision, not all significant bits are printed by default. The maximum number of digits when printing a PARI real can be set using the methods `Pari.set_real_precision_bits()` or `Pari.set_real_precision()`.

We create a very precise approximation of pi and see how it is printed in PARI:

```
sage: pi = pari(RealField(1000).pi())
```

The default precision is 15 digits:

```
sage: pi
3.14159265358979
```

With a different precision:

```
sage: _ = pari.set_real_precision(50)
sage: pi
3.1415926535897932384626433832795028841971693993751
```

Back to the default:

```
sage: _ = pari.set_real_precision(15)
sage: pi
3.14159265358979
```

Input precision for function calls

When we talk about precision for PARI functions, we need to distinguish three kinds of calls:

1. Using the string interface, for example `pari("sin(1)")`.
2. Using the library interface with exact inputs, for example `pari(1).sin()`.
3. Using the library interface with inexact inputs, for example `pari(1.0).sin()`.

In the first case, the relevant precision is the one set by the methods `Pari.set_real_precision_bits()` or `Pari.set_real_precision()`:

```
sage: pari.set_real_precision_bits(150)
sage: pari("sin(1)")
0.841470984807896506652502321630298999622563061
sage: pari.set_real_precision_bits(53)
sage: pari("sin(1)")
0.841470984807897
```

In the second case, the precision can be given as the argument `precision` in the function call, with a default of 53 bits. The real precision set by `Pari.set_real_precision_bits()` or `Pari.set_real_precision()` is irrelevant.

In these examples, we convert to Sage to ensure that PARI's real precision is not used when printing the numbers. As explained before, this artificially increases the precision to a multiple of the wordsize.

```
sage: s = pari(1).sin(precision=180).sage(); print(s); print(parent(s))
0.841470984807896506652502321630298999622563060798371065673
Real Field with 192 bits of precision
sage: s = pari(1).sin(precision=40).sage(); print(s); print(parent(s))
0.841470984807896507
Real Field with 64 bits of precision
sage: s = pari(1).sin().sage(); print(s); print(parent(s))
0.841470984807896507
Real Field with 64 bits of precision
```

In the third case, the precision is determined only by the inexact inputs and the `precision` argument is ignored:

```
sage: pari(1.0).sin(precision=180).sage()
0.841470984807896507
sage: pari(1.0).sin(precision=40).sage()
0.841470984807896507
sage: pari(RealField(100).one()).sin().sage()
0.84147098480789650665250232163029899962
```

Elliptic curve functions

An elliptic curve given with exact a -invariants is considered an exact object. Therefore, you should set the precision for each method call individually:

```
sage: e = pari([0,0,0,-82,0]).ellinit()
sage: eta1 = e.elleta(precision=50)[0]
sage: eta1.sage()
3.6054636014326520859158205642077267748 # 64-bit
3.605463601432652085915820564 # 32-bit
sage: eta1 = e.elleta(precision=150)[0]
```

(continues on next page)

(continued from previous page)

```
sage: eta1.sage()
3.605463601432652085915820564207726774810268996598024745444380641429820491740 # 64-bit
3.60546360143265208591582056420772677481026899659802474544 # 32-bit
```

14.2 Convert PARI objects to Sage types

`sage.libs.pari.convert_sage.gen_to_sage(z, locals=None)`

Convert a PARI gen to a Sage/Python object.

INPUT:

- `z` – PARI gen
- `locals` – (optional) dictionary used in fallback cases that involve `sage_eval()`

OUTPUT: one of the following depending on the PARI type of `z`

- a `Integer` if `z` is an integer (type `t_INT`)
- a `Rational` if `z` is a rational (type `t_FRAC`)
- a `RealNumber` if `z` is a real number (type `t_REAL`). The precision will be equivalent.
- a `NumberFieldElement_quadratic` or a `ComplexNumber` if `z` is a complex number (type `t_COMPLEX`). The former is used when the real and imaginary parts are integers or rationals and the latter when they are floating point numbers. In that case The precision will be the maximal precision of the real and imaginary parts.
- a Python list if `z` is a vector or a list (type `t_VEC`, `t_COL`)
- a Python string if `z` is a string (type `t_STR`)
- a Python list of Python integers if `z` is a small vector (type `t_VECSMALL`)
- a matrix if `z` is a matrix (type `t_MAT`)
- a p -adic element (type `t_PADIC`)
- a `Infinity` if `z` is an infinity (type `t_INF`)

EXAMPLES:

```
sage: from sage.libs.pari.convert_sage import gen_to_sage
```

Converting an integer:

```
sage: z = pari('12'); z
12
sage: z.type()
't_INT'
sage: a = gen_to_sage(z); a
12
sage: a.parent()
Integer Ring

sage: gen_to_sage(pari('7^42'))
311973482284542371301330321821976049
```

Converting a rational number:

(continued from previous page)

```

sage: a = gen_to_sage(z); a #_
↳needs sage.rings.real_mpfr
1.0000000000000000 + 2.0000000000000000*I
sage: a.parent() #_
↳needs sage.rings.real_mpfr
Complex Field with 64 bits of precision

sage: z = pari('1 + 1.0*I'); z
1 + 1.0000000000000000*I
sage: a = gen_to_sage(z); a #_
↳needs sage.rings.real_mpfr
1.0000000000000000 + 1.0000000000000000*I
sage: a.parent() #_
↳needs sage.rings.real_mpfr
Complex Field with 64 bits of precision

sage: z = pari('1.0 + 1*I'); z
1.0000000000000000 + I
sage: a = gen_to_sage(z); a #_
↳needs sage.rings.real_mpfr
1.0000000000000000 + 1.0000000000000000*I
sage: a.parent() #_
↳needs sage.rings.real_mpfr
Complex Field with 64 bits of precision

```

Converting polynomials:

```

sage: f = pari('(2/3)*x^3 + x - 5/7 + y')
sage: f.type()
't_POL'

sage: R.<x,y> = QQ[]
sage: gen_to_sage(f, {'x': x, 'y': y})
2/3*x^3 + x + y - 5/7
sage: parent(gen_to_sage(f, {'x': x, 'y': y}))
Multivariate Polynomial Ring in x, y over Rational Field

sage: # needs sage.symbolic
sage: x,y = SR.var('x,y')
sage: gen_to_sage(f, {'x': x, 'y': y})
2/3*x^3 + x + y - 5/7
sage: parent(gen_to_sage(f, {'x': x, 'y': y}))
Symbolic Ring

sage: gen_to_sage(f)
Traceback (most recent call last):
...
NameError: name 'x' is not defined

```

Converting vectors:

```

sage: # needs sage.rings.number_field sage.rings.real_mpfr
sage: z1 = pari('[-3, 2.1, 1+I]'); z1
[-3, 2.1000000000000000, 1 + I]
sage: z2 = pari('[1.0*I, [1,2]]~'); z2
[1.0000000000000000*I, [1, 2]]~
sage: z1.type(), z2.type()

```

(continues on next page)

(continued from previous page)

```

('t_VEC', 't_COL')
sage: a1 = gen_to_sage(z1)
sage: a2 = gen_to_sage(z2)
sage: type(a1), type(a2)
(<... 'list'>, <... 'list'>)
sage: [parent(b) for b in a1]
[Integer Ring,
 Real Field with 64 bits of precision,
 Number Field in i with defining polynomial x^2 + 1 with i = 1*I]
sage: [parent(b) for b in a2]
[Complex Field with 64 bits of precision, <... 'list'>]

sage: z = pari('Vecsmall([1,2,3,4])')
sage: z.type()
't_VECSMALL'
sage: a = gen_to_sage(z); a
[1, 2, 3, 4]
sage: type(a)
<... 'list'>
sage: [parent(b) for b in a]
[<... 'int'>, <... 'int'>, <... 'int'>, <... 'int'>]

```

Matrices:

```

sage: z = pari('[1,2;3,4]')
sage: z.type()
't_MAT'

sage: # needs sage.modules
sage: a = gen_to_sage(z); a
[1 2]
[3 4]
sage: a.parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring

```

Conversion of p -adics:

```

sage: # needs sage.rings.padics
sage: z = pari('569 + O(7^8)'); z
2 + 4*7 + 4*7^2 + 7^3 + O(7^8)
sage: a = gen_to_sage(z); a
2 + 4*7 + 4*7^2 + 7^3 + O(7^8)
sage: a.parent()
7-adic Field with capped relative precision 8

```

Conversion of infinities:

```

sage: gen_to_sage(pari('oo'))
+Infinity
sage: gen_to_sage(pari('-oo'))
-Infinity

```

Conversion of strings:

```

sage: s = pari('"foo"').sage(); s
'foo'

```

(continues on next page)

(continued from previous page)

```
sage: type(s)
<class 'str'>
```

```
sage.libs.pari.convert_sage.new_gen_from_integer(self)
```

```
sage.libs.pari.convert_sage.new_gen_from_rational(self)
```

```
sage.libs.pari.convert_sage.pari_divisors_small(self)
```

Return the list of divisors of this number using PARI `divisorsu`.

See also

This method is better used through `sage.rings.integer.Integer.divisors()`.

EXAMPLES:

```
sage: from sage.libs.pari.convert_sage import pari_divisors_small
sage: pari_divisors_small(4)
[1, 2, 4]
```

The integer must fit into an unsigned long:

```
sage: pari_divisors_small(-4)
Traceback (most recent call last):
...
AssertionError
sage: pari_divisors_small(2**65)
Traceback (most recent call last):
...
AssertionError
```

```
sage.libs.pari.convert_sage.pari_is_prime(p)
```

Return whether `p` is a prime.

The caller must ensure that `p.value` fits in a long.

EXAMPLES:

```
sage: from sage.libs.pari.convert_sage import pari_is_prime
sage: pari_is_prime(2)
True
sage: pari_is_prime(3)
True
sage: pari_is_prime(1)
False
sage: pari_is_prime(4)
False
```

Its recommended to use `sage.rings.integer.Integer.is_prime()`, which checks overflow. The following is incorrect, because the number does not fit into a long:

```
sage: pari_is_prime(2**64 + 2)
True
```

`sage.libs.pari.convert_sage.pari_is_prime_power(q, get_data)`

Return whether `q` is a prime power.

The caller must ensure that `q.value` fits in a long.

OUTPUT:

If `get_data` return a tuple of the prime and the exponent. Otherwise return a boolean.

EXAMPLES:

```
sage: from sage.libs.pari.convert_sage import pari_is_prime_power
sage: pari_is_prime_power(2, False)
True
sage: pari_is_prime_power(2, True)
(2, 1)
sage: pari_is_prime_power(4, False)
True
sage: pari_is_prime_power(4, True)
(2, 2)
sage: pari_is_prime_power(6, False)
False
sage: pari_is_prime_power(6, True)
(6, 0)
```

It's recommended to use `sage.rings.integer.Integer.is_prime_power()`, which checks overflow. The following is incorrect, because the number does not fit into a long:

```
sage: pari_is_prime_power(2**64 + 2, False)
True
```

`sage.libs.pari.convert_sage.pari_maxprime()`

Return to which limit PARI has computed the primes.

EXAMPLES:

```
sage: from sage.libs.pari.convert_sage import pari_maxprime
sage: a = pari_maxprime()
sage: res = prime_range(2, 2*a)
sage: b = pari_maxprime()
sage: b >= 2*a
True
```

`sage.libs.pari.convert_sage.pari_prime_range(c_start, c_stop, py_ints=False)`

Return a list of all primes between `start` and `stop - 1`, inclusive.

See also

`sage.rings.fast_arith.prime_range()`

`sage.libs.pari.convert_sage.set_integer_from_gen(self, x)`

EXAMPLES:

```
sage: [Integer(pari(x)) for x in [1, 2^60, 2., GF(3)(1), GF(9,'a')(2)]] #_
↳needs sage.rings.finite_rings
[1, 1152921504606846976, 2, 1, 2]
sage: Integer(pari(2.1)) # indirect doctest
```

(continues on next page)

(continued from previous page)

```
Traceback (most recent call last):
...
TypeError: Attempt to coerce non-integral real number to an Integer
```

```
sage.libs.pari.convert_sage.set_rational_from_gen(self, x)
```

EXAMPLES:

```
sage: [Rational(pari(x)) for x in [1, 1/2, 2^60, 2., GF(3)(1), GF(9,'a')(2)]] #_
↳needs sage.rings.finite_rings
[1, 1/2, 1152921504606846976, 2, 1, 2]
sage: Rational(pari(2.1)) # indirect doctest
Traceback (most recent call last):
...
TypeError: Attempt to coerce non-integral real number to an Integer
```

14.3 Ring of pari objects

AUTHORS:

- William Stein (2004): Initial version.
- Simon King (2011-08-24): Use UniqueRepresentation, element_class and proper initialisation of elements.

```
class sage.rings.pari_ring.Pari(x, parent=None)
```

Bases: `RingElement`

Element of Pari pseudo-ring.

```
class sage.rings.pari_ring.PariRing
```

Bases: `Singleton, Parent`

EXAMPLES:

```
sage: R = PariRing(); R
Pseudoring of all PARI objects.
sage: loads(R.dumps()) is R
True
```

Element

alias of `Pari`

characteristic()

is_field (*proof=True*)

random_element (*x=None, y=None, distribution=None*)

Return a random integer in Pari.

Note

The given arguments are passed to `ZZ.random_element(...)`.

INPUT:

- x, y – optional integers, that are lower and upper bound for the result. If only x is provided, then the result is between 0 and $x - 1$, inclusive. If both are provided, then the result is between x and $y - 1$, inclusive.
- `distribution` – (optional) string, so that ZZ can make sense of it as a probability distribution

EXAMPLES:

```
sage: R = PariRing()
sage: R.random_element().parent() is R
True
sage: R(5) <= R.random_element(5,13) < R(13)
True
sage: R.random_element(distribution='1/n').parent() is R
True
```

zeta()

Return -1.

EXAMPLES:

```
sage: R = PariRing()
sage: R.zeta()
-1
```


SYMMETRICA

15.1 Symmetrica library

`sage.libs.symmetrica.symmetrica.bdg_symmetrica(part, perm)`

Calculate the irreducible matrix representation $D^{\text{part}}(\text{perm})$, whose entries are of integral numbers.

REFERENCE: H. Boerner:

Darstellungen von Gruppen, Springer 1955. pp. 104-107.

`sage.libs.symmetrica.symmetrica.chartafel_symmetrica(n)`

You enter the degree of the symmetric group, as INTEGER object and the result is a MATRIX object: the character table of the symmetric group of the given degree.

EXAMPLES:

```
sage: symmetrica.chartafel(3)
[ 1  1  1]
[-1  0  2]
[ 1 -1  1]
sage: symmetrica.chartafel(4)
[ 1  1  1  1  1]
[-1  0 -1  1  3]
[ 0 -1  2  0  2]
[ 1  0 -1 -1  3]
[-1  1  1 -1  1]
```

`sage.libs.symmetrica.symmetrica.charvalue_symmetrica(irred, cls, table=None)`

You enter a PARTITION object part, labelling the irreducible character, you enter a PARTITION object class, labeling the class or class may be a PERMUTATION object, then result becomes the value of that character on that class or permutation. Note that the table may be NULL, in which case the value is computed, or it may be taken from a precalculated character table.

FIXME: add table parameter

EXAMPLES:

```
sage: n = 3
sage: m = matrix([[symmetrica.charvalue(irred, cls) for cls in Partitions(n)] for
↪irred in Partitions(n)]); m
[ 1  1  1]
[-1  0  2]
[ 1 -1  1]
sage: m == symmetrica.chartafel(n)
True
```

(continues on next page)

(continued from previous page)

```

sage: n = 4
sage: m = matrix([[symmetrica.charvalue(irred, cls) for cls in Partitions(n)] for
↳irred in Partitions(n)])
sage: m == symmetrica.chartafel(n)
True

```

```

sage.libs.symmetrica.symmetrica.compute_elmsym_with_alphabet_symmetrica(n, length,
                                                                    alpha-
                                                                    bet='x')

```

Compute the expansion of a elementary symmetric function labeled by a INTEGER number as a POLYNOM erg. The object number may also be a PARTITION or a ELM_SYM object. The INTEGER length specifies the length of the alphabet. Both routines are the same.

EXAMPLES:

```

sage: a = symmetrica.compute_elmsym_with_alphabet(2,2); a
x0*x1
sage: a.parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring
sage: a = symmetrica.compute_elmsym_with_alphabet([2],2); a
x0*x1
sage: symmetrica.compute_elmsym_with_alphabet(3,2)
0
sage: symmetrica.compute_elmsym_with_alphabet([3,2,1],2)
0

```

```

sage.libs.symmetrica.symmetrica.compute_homsym_with_alphabet_symmetrica(n, length,
                                                                    alpha-
                                                                    bet='x')

```

Compute the expansion of a homogeneous(=complete) symmetric function labeled by a INTEGER number as a POLYNOM erg. The object number may also be a PARTITION or a HOM_SYM object. The INTEGER laenge specifies the length of the alphabet. Both routines are the same.

EXAMPLES:

```

sage: symmetrica.compute_homsym_with_alphabet(3,1,'x')
x^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],1,'x')
x^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],2,'x')
x0^3 + 2*x0^2*x1 + 2*x0*x1^2 + x1^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],2,'a,b')
a^3 + 2*a^2*b + 2*a*b^2 + b^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],2,'x').parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring

```

```

sage.libs.symmetrica.symmetrica.compute_monomial_with_alphabet_symmetrica(n,
                                                                    length,
                                                                    alpha-
                                                                    bet='x')

```

Compute the expansion of a monomial symmetric function labeled by a PARTITION number as a POLYNOM erg. The INTEGER laenge specifies the length of the alphabet.

EXAMPLES:


```

sage: symmetrica.compute_monomial_with_alphabet([2,1],2,'x')
x0^2*x1 + x0*x1^2
sage: symmetrica.compute_monomial_with_alphabet([1,1,1],2,'x')
0
sage: symmetrica.compute_monomial_with_alphabet(2,2,'x')
x0^2 + x1^2
sage: symmetrica.compute_monomial_with_alphabet(2,2,'a,b')
a^2 + b^2
sage: symmetrica.compute_monomial_with_alphabet(2,2,'x').parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring

```

```

sage.libs.symmetrica.symmetrica.compute_powsym_with_alphabet_symmetrica(n, length,
                                                                           alpha-
                                                                           bet='x')

```

Compute the expansion of a power symmetric function labeled by a INTEGER label or by a PARTITION label or a POW_SYM label as a POLYNOM erg. The INTEGER laenge specifies the length of the alphabet.

EXAMPLES:

```

sage: symmetrica.compute_powsym_with_alphabet(2,2,'x')
x0^2 + x1^2
sage: symmetrica.compute_powsym_with_alphabet(2,2,'x').parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring
sage: symmetrica.compute_powsym_with_alphabet([2],2,'x')
x0^2 + x1^2
sage: symmetrica.compute_powsym_with_alphabet([2],2,'a,b')
a^2 + b^2
sage: symmetrica.compute_powsym_with_alphabet([2,1],2,'a,b')
a^3 + a^2*b + a*b^2 + b^3

```

```

sage.libs.symmetrica.symmetrica.compute_schur_with_alphabet_det_symmetrica(part,
                                                                              length,
                                                                              al-
                                                                              pha-
                                                                              bet='x')

```

EXAMPLES:

```

sage: symmetrica.compute_schur_with_alphabet_det(2,2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet_det([2],2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet_det(Partition([2]),2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet_det(Partition([2]),2,'y')
y0^2 + y0*y1 + y1^2
sage: symmetrica.compute_schur_with_alphabet_det(Partition([2]),2,'a,b')
a^2 + a*b + b^2

```

```

sage.libs.symmetrica.symmetrica.compute_schur_with_alphabet_symmetrica(part,
                                                                              length,
                                                                              alpha-
                                                                              bet='x')

```

Compute the expansion of a schurfunction labeled by a partition PART as a POLYNOM erg. The INTEGER length specifies the length of the alphabet.

EXAMPLES:

```

sage: symmetrica.compute_schur_with_alphabet(2,2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet([2],2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet(Partition([2]),2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet(Partition([2]),2,'y')
y0^2 + y0*y1 + y1^2
sage: symmetrica.compute_schur_with_alphabet(Partition([2]),2,'a,b')
a^2 + a*b + b^2
sage: symmetrica.compute_schur_with_alphabet([2,1],1,'x')
0

```

sage.libs.symmetrica.symmetrica.**dimension_schur_symmetrica**(*s*)

You enter a SCHUR object *a*, and the result is the dimension of the corresponding representation of the symmetric group *sn*.

sage.libs.symmetrica.symmetrica.**dimension_symmetrization_symmetrica**(*n, part*)

Compute the dimension of the degree of a irreducible representation of the GL_n , *n* is a INTEGER object, labeled by the PARTITION object *a*.

sage.libs.symmetrica.symmetrica.**divdiff_perm_schubert_symmetrica**(*perm, a*)

Return the result of applying the divided difference operator δ_i to *a* where *a* is either a permutation or a Schubert polynomial over \mathbf{Q} .

EXAMPLES:

```

sage: symmetrica.divdiff_perm_schubert([2,3,1],[3,2,1])
X[2, 1]
sage: symmetrica.divdiff_perm_schubert([3,1,2],[3,2,1])
X[1, 3, 2]
sage: symmetrica.divdiff_perm_schubert([3,2,4,1],[3,2,1])
Traceback (most recent call last):
...
ValueError: cannot apply \delta_{[3, 2, 4, 1]} to a (= [3, 2, 1])

```

sage.libs.symmetrica.symmetrica.**divdiff_schubert_symmetrica**(*i, a*)

Return the result of applying the divided difference operator δ_i to *a* where *a* is either a permutation or a Schubert polynomial over \mathbf{Q} .

EXAMPLES:

```

sage: symmetrica.divdiff_schubert(1,[3,2,1])
X[2, 3, 1]
sage: symmetrica.divdiff_schubert(2,[3,2,1])
X[3, 1, 2]
sage: symmetrica.divdiff_schubert(3,[3,2,1])
Traceback (most recent call last):
...
ValueError: cannot apply \delta_{3} to a (= [3, 2, 1])

```

sage.libs.symmetrica.symmetrica.**gupta_nm_symmetrica**(*n, m*)

This routine computes the number of partitions of *n* with maximal part *m*. The result is erg. The input *n,m* must be INTEGER objects. The result is freed first to an empty object. The result must be a different from *m* and *n*.

sage.libs.symmetrica.symmetrica.**gupta_tafel_symmetrica**(*max*)

It computes the table of the above values. The entry *n,m* is the result of `gupta_nm`. *mat* is freed first. *max* must be an INTEGER object, it is the maximum weight for the partitions. *max* must be different from result.

`sage.libs.symmetrica.symmetrica.hall_littlewood_symmetrica` (*part*)

Compute the so called Hall Littlewood Polynomials, i.e. a SCHUR object, whose coefficient are polynomials in one variable. The method, which is used for the computation is described in the paper: A.O. Morris The Characters of the group $GL(n,q)$ Math Zeitschr 81, 112-123 (1963)

`sage.libs.symmetrica.symmetrica.kostka_number_symmetrica` (*shape, content*)

Compute the kostkanumber, i.e. the number of tableaux of given shape, which is a PARTITION object, and of given content, which also is a PARTITION object, or a VECTOR object with INTEGER entries. The result is an INTEGER object, which is freed to an empty object at the beginning. The shape could also be a SKEWPARTITION object, then we compute the number of skewtableaux of the given shape.

EXAMPLES:

```
sage: symmetrica.kostka_number([2,1],[1,1,1])
2
sage: symmetrica.kostka_number([1,1,1],[1,1,1])
1
sage: symmetrica.kostka_number([3],[1,1,1])
1
```

`sage.libs.symmetrica.symmetrica.kostka_tab_symmetrica` (*shape, content*)

Compute the list of tableaux of given shape and content. shape is a PARTITION object or a SKEWPARTITION object and content is a PARTITION object or a VECTOR object with INTEGER entries, the result becomes a LIST object whose entries are the computed TABLEAUX object.

EXAMPLES:

```
sage: symmetrica.kostka_tab([3],[1,1,1])
[[[1, 2, 3]]]
sage: symmetrica.kostka_tab([2,1],[1,1,1])
[[[1, 2], [3]], [[1, 3], [2]]]
sage: symmetrica.kostka_tab([1,1,1],[1,1,1])
[[[1], [2], [3]]]
sage: symmetrica.kostka_tab([[2,2,1],[1,1]],[1,1,1])
[[[None, 1], [None, 2], [3]],
 [[None, 1], [None, 3], [2]],
 [[None, 2], [None, 3], [1]]]
sage: symmetrica.kostka_tab([[2,2],[1]],[1,1,1])
[[[None, 1], [2, 3]], [[None, 2], [1, 3]]]
```

`sage.libs.symmetrica.symmetrica.kostka_tafel_symmetrica` (*n*)

Return the table of Kostka numbers of weight *n*.

EXAMPLES:

```
sage: symmetrica.kostka_tafel(1)
[1]

sage: symmetrica.kostka_tafel(2)
[1 0]
[1 1]

sage: symmetrica.kostka_tafel(3)
[1 0 0]
[1 1 0]
[1 2 1]
```

(continues on next page)

(continued from previous page)

```

sage: symmetrica.kostka_tafel(4)
[1 0 0 0 0]
[1 1 0 0 0]
[1 1 1 0 0]
[1 2 1 1 0]
[1 3 2 3 1]

sage: symmetrica.kostka_tafel(5)
[1 0 0 0 0 0]
[1 1 0 0 0 0]
[1 1 1 0 0 0]
[1 2 1 1 0 0]
[1 2 2 1 1 0]
[1 3 3 3 2 1]
[1 4 5 6 5 4 1]

```

sage.libs.symmetrica.symmetrica.kranztafel_symmetrica(*a*, *b*)

You enter the INTEGER objects, say *a* and *b*, and *res* becomes a MATRIX object, the charactertable of $S_b \wr S_a$, *co* becomes a VECTOR object of classorders and *cl* becomes a VECTOR object of the classlabels.

EXAMPLES:

```

sage: (a,b,c) = symmetrica.kranztafel(2,2)
sage: a
[ 1 -1  1 -1  1]
[ 1  1  1  1  1]
[-1  1  1 -1  1]
[ 0  0  2  0 -2]
[-1 -1  1  1  1]
sage: b
[2, 2, 1, 2, 1]
sage: for m in c: print(m)
[0 0]
[0 1]
[0 0]
[1 0]
[0 2]
[0 0]
[1 1]
[0 0]
[2 0]
[0 0]

```

sage.libs.symmetrica.symmetrica.mult_monomial_monomial_symmetrica(*m1*, *m2*)

sage.libs.symmetrica.symmetrica.mult_schubert_schubert_symmetrica(*a*, *b*)

Multiply the Schubert polynomials *a* and *b*.

EXAMPLES:

```

sage: symmetrica.mult_schubert_schubert([3,2,1], [3,2,1])
X[5, 3, 1, 2, 4]

```

sage.libs.symmetrica.symmetrica.mult_schubert_variable_symmetrica(*a*, *i*)

Return the product of *a* and x_i . Note that indexing with *i* starts at 1.

EXAMPLES:

```
sage: symmetrica.mult_schubert_variable([3,2,1], 2)
X[3, 2, 4, 1]
sage: symmetrica.mult_schubert_variable([3,2,1], 4)
X[3, 2, 1, 4, 6, 5] - X[3, 2, 1, 5, 4]
```

sage.libs.symmetrica.symmetrica.**mult_schur_schur_symmetrica** (*s1, s2*)

sage.libs.symmetrica.symmetrica.**ndg_symmetrica** (*part, perm*)

sage.libs.symmetrica.symmetrica.**newtrans_symmetrica** (*perm*)

Compute the decomposition of a schubertpolynomial labeled by the permutation perm, as a sum of Schurfunction.

FIXME!

sage.libs.symmetrica.symmetrica.**odd_to_strict_part_symmetrica** (*part*)

Implement the bijection between partitions with odd parts and strict partitions. input is a VECTOR type partition, the result is a partition of the same weight with different parts.

sage.libs.symmetrica.symmetrica.**odg_symmetrica** (*part, perm*)

Calculate the irreducible matrix representation $D^{\text{part}}(\text{perm})$, which consists of real numbers.

REFERENCE: G. James/ A. Kerber:

Representation Theory of the Symmetric Group. Addison/Wesley 1981. pp. 127-129.

sage.libs.symmetrica.symmetrica.**outerproduct_schur_symmetrica** (*parta, partb*)

You enter two PARTITION objects, and the result is a SCHUR object, which is the expansion of the product of the two schurfunctions, labeled by the two PARTITION objects parta and partb. Of course this can also be interpreted as the decomposition of the outer tensor product of two irreducible representations of the symmetric group.

EXAMPLES:

```
sage: symmetrica.outerproduct_schur([2],[2])
s[2, 2] + s[3, 1] + s[4]
```

sage.libs.symmetrica.symmetrica.**part_part_skewschur_symmetrica** (*outer, inner*)

Return the skew Schur function $s_{\{\text{outer/inner}\}}$.

EXAMPLES:

```
sage: symmetrica.part_part_skewschur([3,2,1],[2,1])
s[1, 1, 1] + 2*s[2, 1] + s[3]
```

sage.libs.symmetrica.symmetrica.**plethysm_symmetrica** (*outer, inner*)

sage.libs.symmetrica.symmetrica.**q_core_symmetrica** (*part, d*)

Compute the q-core of a PARTITION object part. This is the remaining partition (=res) after removing of all hooks of length d (= INTEGER object). The result may be an empty object, if the whole partition disappears.

sage.libs.symmetrica.symmetrica.**random_partition_symmetrica** (*n*)

Return a random partition p of the entered weight w.

w must be an INTEGER object, p becomes a PARTITION object. Type of partition is VECTOR . It uses the algorithm of Nijenhuis and Wilf, p.76

sage.libs.symmetrica.symmetrica.**scalarproduct_schubert_symmetrica** (*a, b*)

EXAMPLES:

```

sage: symmetrica.scalarproduct_schubert([3,2,1], [3,2,1])
X[1, 3, 5, 2, 4]
sage: symmetrica.scalarproduct_schubert([3,2,1], [2,1,3])
X[1, 2, 4, 3]

```

sage.libs.symmetrica.symmetrica.**scalarproduct_schur_symmetrica**(*s1, s2*)

sage.libs.symmetrica.symmetrica.**schur_schur_plet_symmetrica**(*outer, inner*)

sage.libs.symmetrica.symmetrica.**sdg_symmetrica**(*part, perm*)

Calculate the irreducible matrix representation $D^{\text{part}}(\text{perm})$, which consists of rational numbers.

REFERENCE: G. James/ A. Kerber:

Representation Theory of the Symmetric Group. Addison/Wesley 1981. pp. 124-126.

sage.libs.symmetrica.symmetrica.**specht_dg_symmetrica**(*part, perm*)

sage.libs.symmetrica.symmetrica.**start**()

sage.libs.symmetrica.symmetrica.**strict_to_odd_part_symmetrica**(*part*)

Implement the bijection between strict partitions and partitions with odd parts. input is a VECTOR type partition, the result is a partition of the same weight with only odd parts.

sage.libs.symmetrica.symmetrica.**t_ELMSYM_HOMSYM_symmetrica**(*elmsym*)

sage.libs.symmetrica.symmetrica.**t_ELMSYM_MONOMIAL_symmetrica**(*elmsym*)

sage.libs.symmetrica.symmetrica.**t_ELMSYM_POWSYM_symmetrica**(*elmsym*)

sage.libs.symmetrica.symmetrica.**t_ELMSYM_SCHUR_symmetrica**(*elmsym*)

sage.libs.symmetrica.symmetrica.**t_HOMSYM_ELMSYM_symmetrica**(*homsym*)

sage.libs.symmetrica.symmetrica.**t_HOMSYM_MONOMIAL_symmetrica**(*homsym*)

sage.libs.symmetrica.symmetrica.**t_HOMSYM_POWSYM_symmetrica**(*homsym*)

sage.libs.symmetrica.symmetrica.**t_HOMSYM_SCHUR_symmetrica**(*homsym*)

sage.libs.symmetrica.symmetrica.**t_MONOMIAL_ELMSYM_symmetrica**(*monomial*)

sage.libs.symmetrica.symmetrica.**t_MONOMIAL_HOMSYM_symmetrica**(*monomial*)

sage.libs.symmetrica.symmetrica.**t_MONOMIAL_POWSYM_symmetrica**(*monomial*)

sage.libs.symmetrica.symmetrica.**t_MONOMIAL_SCHUR_symmetrica**(*monomial*)

sage.libs.symmetrica.symmetrica.**t_POLYNOM_ELMSYM_symmetrica**(*p*)

Convert a symmetric polynomial with base ring \mathbf{Q} or \mathbf{Z} into a symmetric function in the elementary basis.

sage.libs.symmetrica.symmetrica.**t_POLYNOM_MONOMIAL_symmetrica**(*p*)

Convert a symmetric polynomial with base ring \mathbf{Q} or \mathbf{Z} into a symmetric function in the monomial basis.

sage.libs.symmetrica.symmetrica.**t_POLYNOM_POWER_symmetrica**(*p*)

Convert a symmetric polynomial with base ring \mathbf{Q} or \mathbf{Z} into a symmetric function in the power sum basis.

```
sage.libs.symmetrica.symmetrica.t_POLYNOM_SCHUBERT_symmetrica(a)
```

Convert a multivariate polynomial a to a Schubert polynomial.

EXAMPLES:

```
sage: R.<x1,x2,x3> = QQ[]
sage: w0 = x1^2*x2
sage: symmetrica.t_POLYNOM_SCHUBERT(w0)
X[3, 2, 1]
```

```
sage.libs.symmetrica.symmetrica.t_POLYNOM_SCHUR_symmetrica(p)
```

Convert a symmetric polynomial with base ring \mathbf{Q} or \mathbf{Z} into a symmetric function in the Schur basis.

```
sage.libs.symmetrica.symmetrica.t_POWSYM_ELMSYM_symmetrica(powsym)
```

```
sage.libs.symmetrica.symmetrica.t_POWSYM_HOMSYM_symmetrica(powsym)
```

```
sage.libs.symmetrica.symmetrica.t_POWSYM_MONOMIAL_symmetrica(powsym)
```

```
sage.libs.symmetrica.symmetrica.t_POWSYM_SCHUR_symmetrica(powsym)
```

```
sage.libs.symmetrica.symmetrica.t_SCHUBERT_POLYNOM_symmetrica(a)
```

Convert a Schubert polynomial to a ‘regular’ multivariate polynomial.

EXAMPLES:

```
sage: symmetrica.t_SCHUBERT_POLYNOM([3,2,1])
x0^2*x1
```

```
sage.libs.symmetrica.symmetrica.t_SCHUR_ELMSYM_symmetrica(schur)
```

```
sage.libs.symmetrica.symmetrica.t_SCHUR_HOMSYM_symmetrica(schur)
```

```
sage.libs.symmetrica.symmetrica.t_SCHUR_MONOMIAL_symmetrica(schur)
```

```
sage.libs.symmetrica.symmetrica.t_SCHUR_POWSYM_symmetrica(schur)
```

```
sage.libs.symmetrica.symmetrica.test_integer(x)
```

Test functionality for converting between Sage’s integers and symmetrica’s integers.

EXAMPLES:

```
sage: from sage.libs.symmetrica.symmetrica import test_integer
sage: test_integer(1)
1
sage: test_integer(-1)
-1
sage: test_integer(2^33)
8589934592
sage: test_integer(-2^33)
-8589934592
sage: test_integer(2^100)
1267650600228229401496703205376
sage: test_integer(-2^100)
-1267650600228229401496703205376
sage: for i in range(100):
.....:     if test_integer(2^i) != 2^i:
.....:         print("Failure at {}".format(i))
```


INDICES AND TABLES

- [Index](#)
- [Module Index](#)
- [Search Page](#)

PYTHON MODULE INDEX

I

sage.libs.ecl, 3
sage.libs.eclib.constructor, 36
sage.libs.eclib.homspace, 33
sage.libs.eclib.interface, 11
sage.libs.eclib.mat, 29
sage.libs.eclib.mwrank, 27
sage.libs.eclib.newforms, 31
sage.libs.flint.arith_sage, 41
sage.libs.flint.fmpq_poly_sage, 41
sage.libs.flint.fmpz_poly_sage, 39
sage.libs.flint.qsieve_sage, 43
sage.libs.flint.ulong_extras_sage, 44
sage.libs.gap.context_managers, 83
sage.libs.gap.element, 92
sage.libs.gap.gap_functions, 84
sage.libs.gap.libgap, 85
sage.libs.gap.saved_workspace, 107
sage.libs.gap.test, 91
sage.libs.gap.test_long, 84
sage.libs.gap.util, 84
sage.libs.giac, 45
sage.libs.gsl.array, 51
sage.libs.lcalc.lcalc_Lfunction, 53
sage.libs.libecm, 49
sage.libs.linbox.linbox_flint_inter-
face, 109
sage.libs.lrcalc.lrcalc, 111
sage.libs.mpmath.utils, 119
sage.libs.ntl.all, 125
sage.libs.pari, 127
sage.libs.pari.convert_sage, 130
sage.libs.singular.function, 61
sage.libs.singular.function_factory, 70
sage.libs.singular.groebner_strategy,
79
sage.libs.singular.option, 72
sage.libs.singular.polynomial, 72
sage.libs.singular.ring, 77
sage.libs.singular.singular, 70
sage.libs.symmetrica.symmetrica, 139

r

sage.rings.pari_ring, 136

A

add_scalar() (*sage.libs.eclib.mat.Matrix method*), 29
ainvs() (*sage.libs.eclib.interface.mwrank_EllipticCurve method*), 12
all_singular_poly_wrapper() (*in module sage.libs.singular.function*), 65
all_vectors() (*in module sage.libs.singular.function*), 65
atomp() (*sage.libs.ecl.EclObject method*), 5

B

BaseCallHandler (*class in sage.libs.singular.function*), 62
bdg_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 139
bell_number() (*in module sage.libs.flint.arith_sage*), 41
bernoulli_number() (*in module sage.libs.flint.arith_sage*), 41
bitcount() (*in module sage.libs.mpmath.utils*), 119

C

caar() (*sage.libs.ecl.EclObject method*), 5
cadr() (*sage.libs.ecl.EclObject method*), 5
call() (*in module sage.libs.mpmath.utils*), 119
car() (*sage.libs.ecl.EclObject method*), 6
cdar() (*sage.libs.ecl.EclObject method*), 6
cddr() (*sage.libs.ecl.EclObject method*), 6
cdr() (*sage.libs.ecl.EclObject method*), 7
certain() (*sage.libs.eclib.interface.mwrank_EllipticCurve method*), 12
characteristic() (*sage.libs.singular.function.RingWrap method*), 63
characteristic() (*sage.rings.pari_ring.PariRing method*), 136
characterp() (*sage.libs.ecl.EclObject method*), 7
charpoly() (*sage.libs.eclib.mat.Matrix method*), 29
chartafel_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 139
charvalue_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 139
collect() (*sage.libs.gap.libgap.Gap method*), 88

compute_elmsym_with_alphabet_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 140
compute_homsym_with_alphabet_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 140
compute_monomial_with_alphabet_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 140
compute_powsym_with_alphabet_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 141
compute_rank() (*sage.libs.lcalc.lcalc_Lfunction.Lfunction method*), 53
compute_schur_with_alphabet_det_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 141
compute_schur_with_alphabet_symmetrica() (*in module sage.libs.symmetrica.symmetrica*), 141
conductor() (*sage.libs.eclib.interface.mwrank_EllipticCurve method*), 13
cons() (*sage.libs.ecl.EclObject method*), 7
consp() (*sage.libs.ecl.EclObject method*), 7
Converter (*class in sage.libs.singular.function*), 62
coprod() (*in module sage.libs.lrcalc.lrcalc*), 114
count_GAP_objects() (*sage.libs.gap.libgap.Gap method*), 88
CPS_height_bound() (*sage.libs.eclib.interface.mwrank_EllipticCurve method*), 12
CremonaModularSymbols() (*in module sage.libs.eclib.constructor*), 36
currRing_wrapper() (*in module sage.libs.singular.ring*), 77

D

dedekind_sum() (*in module sage.libs.flint.arith_sage*), 42
deepcopy() (*sage.libs.gap.element.GapElement method*), 92
degree() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly method*), 39

- `derivative()` (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 39
`dimension()` (*sage.libs.eclib.homospace.ModularSymbols* method), 33
`dimension_schur_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 142
`dimension_symmetrization_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 142
`div_rem()` (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 40
`divdiff_perm_schubert_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 142
`divdiff_schubert_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 142
- ## E
- `ecl_eval()` (*in module sage.libs.ecl*), 9
`EclListIterator` (*class in sage.libs.ecl*), 3
`EclObject` (*class in sage.libs.ecl*), 3
`ecmfactor()` (*in module sage.libs.libecm*), 49
`ECModularSymbol` (*class in sage.libs.eclib.newforms*), 31
`Element` (*sage.libs.gap.libgap.Gap* attribute), 88
`Element` (*sage.rings.pari_ring.PariRing* attribute), 136
`euler_number()` (*in module sage.libs.flint.arith_sage*), 42
`eval()` (*sage.libs.ecl.EclObject* method), 8
`eval()` (*sage.libs.gap.libgap.Gap* method), 88
- ## F
- `find_zeros()` (*sage.libs.lcalc.lcalc_Lfunction.Lfunction* method), 53
`find_zeros_via_N()` (*sage.libs.lcalc.lcalc_Lfunction.Lfunction* method), 54
`fixnump()` (*sage.libs.ecl.EclObject* method), 8
`Fmpz_poly` (*class in sage.libs.flint.fmpz_poly_sage*), 39
`from_man_exp()` (*in module sage.libs.mpmath.utils*), 120
`function_factory()` (*sage.libs.gap.libgap.Gap* method), 88
- ## G
- `Gap` (*class in sage.libs.gap.libgap*), 87
`GapElement` (*class in sage.libs.gap.element*), 92
`GapElement_Boolean` (*class in sage.libs.gap.element*), 95
`GapElement_Cyclotomic` (*class in sage.libs.gap.element*), 96
`GapElement_FiniteField` (*class in sage.libs.gap.element*), 96
`GapElement_Float` (*class in sage.libs.gap.element*), 97
`GapElement_Function` (*class in sage.libs.gap.element*), 98
`GapElement_Integer` (*class in sage.libs.gap.element*), 98
`GapElement_IntegerMod` (*class in sage.libs.gap.element*), 99
`GapElement_List` (*class in sage.libs.gap.element*), 99
`GapElement_MethodProxy` (*class in sage.libs.gap.element*), 101
`GapElement_Permutation` (*class in sage.libs.gap.element*), 102
`GapElement_Rational` (*class in sage.libs.gap.element*), 102
`GapElement_Record` (*class in sage.libs.gap.element*), 103
`GapElement_RecordIterator` (*class in sage.libs.gap.element*), 104
`GapElement_Ring` (*class in sage.libs.gap.element*), 104
`GapElement_String` (*class in sage.libs.gap.element*), 106
`GAPError`, 84
`gen_to_sage()` (*in module sage.libs.pari.convert_sage*), 130
`gens()` (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 13
`get_global()` (*sage.libs.gap.libgap.Gap* method), 89
`get_owned_objects()` (*in module sage.libs.gap.util*), 84
`get_precision()` (*in module sage.libs.eclib.mwrank*), 27
`get_printlevel()` (*in module sage.libs.singular.function*), 65
`get_resource()` (*in module sage.libs.singular.singular*), 70
`GiacSettingsDefaultContext` (*class in sage.libs.giac*), 45
`global_context()` (*sage.libs.gap.libgap.Gap* method), 89
`GlobalVariableContext` (*class in sage.libs.gap.context_managers*), 83
`groebner_basis()` (*in module sage.libs.giac*), 45
`GroebnerStrategy` (*class in sage.libs.singular.groebner_strategy*), 79
`GSLDoubleArray` (*class in sage.libs.gsl.array*), 51
`gupta_nm_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 142
`gupta_tafel_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 142
- ## H
- `hall_littlewood_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 143
`hardy_z_function()` (*sage.libs.lcalc.lcalc_Lfunction.Lfunction* method), 55
`harmonic_number()` (*in module sage.libs.flint.arith_sage*), 42

- hecke_matrix() (*sage.libs.eclib.homspace.ModularSymbols* method), 33
- ## I
- ideal() (*sage.libs.singular.groebner_strategy.GroebnerStrategy* method), 79
- ideal() (*sage.libs.singular.groebner_strategy.NCGroebnerStrategy* method), 80
- init_ecl() (in module *sage.libs.ecl*), 9
- initprimes() (in module *sage.libs.eclib.mwrank*), 27
- is_bool() (*sage.libs.gap.element.GapElement* method), 93
- is_C_int() (*sage.libs.gap.element.GapElement_Integer* method), 98
- is_commutative() (*sage.libs.singular.function.RingWrap* method), 63
- is_cuspidal() (*sage.libs.eclib.homspace.ModularSymbols* method), 34
- is_field() (*sage.rings.pari_ring.PariRing* method), 136
- is_function() (*sage.libs.gap.element.GapElement* method), 93
- is_list() (*sage.libs.gap.element.GapElement* method), 93
- is_permutation() (*sage.libs.gap.element.GapElement* method), 93
- is_record() (*sage.libs.gap.element.GapElement* method), 93
- is_sage_wrapper_for_singular_ring() (in module *sage.libs.singular.function*), 65
- is_singular_poly_wrapper() (in module *sage.libs.singular.function*), 66
- is_string() (*sage.libs.gap.element.GapElement* method), 94
- isogeny_class() (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 13
- isqrt() (in module *sage.libs.mpmath.utils*), 121
- ## K
- KernelCallHandler (class in *sage.libs.singular.function*), 62
- kostka_number_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 143
- kostka_tab_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 143
- kostka_tafel_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 143
- kranztafel_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 144
- ## L
- left_shift() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 40
- level() (*sage.libs.eclib.homspace.ModularSymbols* method), 34
- Lfunction (class in *sage.libs.lcalc.lcalc_Lfunction*), 53
- Lfunction_C (class in *sage.libs.lcalc.lcalc_Lfunction*), 56
- Lfunction_D (class in *sage.libs.lcalc.lcalc_Lfunction*), 57
- Lfunction_from_character() (in module *sage.libs.lcalc.lcalc_Lfunction*), 58
- Lfunction_from_elliptic_curve() (in module *sage.libs.lcalc.lcalc_Lfunction*), 59
- Lfunction_I (class in *sage.libs.lcalc.lcalc_Lfunction*), 58
- Lfunction_Zeta (class in *sage.libs.lcalc.lcalc_Lfunction*), 58
- lib() (in module *sage.libs.singular.function*), 66
- LibraryCallHandler (class in *sage.libs.singular.function*), 62
- LibSingularOptions (class in *sage.libs.singular.option*), 73
- LibSingularOptions_abstract (class in *sage.libs.singular.option*), 76
- LibSingularOptionsContext (class in *sage.libs.singular.option*), 75
- LibSingularVerboseOptions (class in *sage.libs.singular.option*), 76
- lift() (*sage.libs.gap.element.GapElement_FiniteField* method), 96
- lift() (*sage.libs.gap.element.GapElement_IntegerMod* method), 99
- list() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 40
- list_of_functions() (in module *sage.libs.singular.function*), 66
- listp() (*sage.libs.ecl.EclObject* method), 8
- load() (*sage.libs.singular.option.LibSingularOptions_abstract* method), 76
- load_package() (*sage.libs.gap.libgap.Gap* method), 89
- local_giacsettings() (in module *sage.libs.giac*), 47
- lrcoef() (in module *sage.libs.lrcalc.lrcalc*), 114
- lrcoef_unsafe() (in module *sage.libs.lrcalc.lrcalc*), 114
- lrskew() (in module *sage.libs.lrcalc.lrcalc*), 115
- ## M
- Matrix (class in *sage.libs.eclib.mat*), 29
- matrix() (*sage.libs.gap.element.GapElement_List* method), 100
- MatrixFactory (class in *sage.libs.eclib.mat*), 31
- ModularSymbols (class in *sage.libs.eclib.homspace*), 33
- module
sage.libs.ecl, 3

sage.libs.eclib.constructor, 36
 sage.libs.eclib.homspace, 33
 sage.libs.eclib.interface, 11
 sage.libs.eclib.mat, 29
 sage.libs.eclib.mwrank, 27
 sage.libs.eclib.newforms, 31
 sage.libs.flint.arith_sage, 41
 sage.libs.flint.fmpq_poly_sage, 41
 sage.libs.flint.fmpz_poly_sage, 39
 sage.libs.flint.qsieve_sage, 43
 sage.libs.flint.ulong_extras_sage, 44
 sage.libs.gap.context_managers, 83
 sage.libs.gap.element, 92
 sage.libs.gap.gap_functions, 84
 sage.libs.gap.libgap, 85
 sage.libs.gap.saved_workspace, 107
 sage.libs.gap.test, 91
 sage.libs.gap.test_long, 84
 sage.libs.gap.util, 84
 sage.libs.giac, 45
 sage.libs.gsl.array, 51
 sage.libs.lcalc.lcalc_Lfunction, 53
 sage.libs.libecm, 49
 sage.libs.linbox.linbox_flint_interface, 109
 sage.libs.lrcalc.lrcalc, 111
 sage.libs.mpmath.utils, 119
 sage.libs.ntl.all, 125
 sage.libs.pari, 127
 sage.libs.pari.convert_sage, 130
 sage.libs.singular.function, 61
 sage.libs.singular.function_factory, 70
 sage.libs.singular.groebner_strategy, 79
 sage.libs.singular.option, 72
 sage.libs.singular.polynomial, 72
 sage.libs.singular.ring, 77
 sage.libs.singular.singular, 70
 sage.libs.symmetrica.symmetrica, 139
 sage.rings.pari_ring, 136
 mpmath_to_sage() (in module sage.libs.mpmath.utils), 121
 mult() (in module sage.libs.lrcalc.lrcalc), 116
 mult_monomial_monomial_symmetrica() (in module sage.libs.symmetrica.symmetrica), 144
 mult_schubert() (in module sage.libs.lrcalc.lrcalc), 117
 mult_schubert_schubert_symmetrica() (in module sage.libs.symmetrica.symmetrica), 144
 mult_schubert_variable_symmetrica() (in module sage.libs.symmetrica.symmetrica), 144

mult_schur_schur_symmetrica() (in module sage.libs.symmetrica.symmetrica), 145
 mwrank_EllipticCurve (class in sage.libs.eclib.interface), 11
 mwrank_MordellWeil (class in sage.libs.eclib.interface), 17

N

n_factor_to_list() (in module sage.libs.flint.ulong_extras_sage), 44
 NCGroebnerStrategy (class in sage.libs.singular.groebner_strategy), 79
 ncols() (sage.libs.eclib.mat.Matrix method), 30
 ndg_symmetrica() (in module sage.libs.symmetrica.symmetrica), 145
 new_gen_from_integer() (in module sage.libs.pari.convert_sage), 134
 new_gen_from_rational() (in module sage.libs.pari.convert_sage), 134
 newtrans_symmetrica() (in module sage.libs.symmetrica.symmetrica), 145
 ngens() (sage.libs.singular.function.RingWrap method), 63
 normal_form() (sage.libs.singular.groebner_strategy.GroebnerStrategy method), 79
 normal_form() (sage.libs.singular.groebner_strategy.NCGroebnerStrategy method), 80
 normalize() (in module sage.libs.mpmath.utils), 122
 npars() (sage.libs.singular.function.RingWrap method), 63
 nrows() (sage.libs.eclib.mat.Matrix method), 30
 nullp() (sage.libs.ecl.EclObject method), 8
 number_of_cusps() (sage.libs.eclib.homspace.ModularSymbols method), 34
 number_of_partitions() (in module sage.libs.flint.arith_sage), 42

O

ObjWrapper (class in sage.libs.gap.util), 84
 odd_to_strict_part_symmetrica() (in module sage.libs.symmetrica.symmetrica), 145
 odg_symmetrica() (in module sage.libs.symmetrica.symmetrica), 145
 one() (sage.libs.gap.libgap.Gap method), 89
 opt (sage.libs.singular.option.LibSingularOptionsContext attribute), 76
 ordering_string() (sage.libs.singular.function.RingWrap method), 63
 outerproduct_schur_symmetrica() (in module sage.libs.symmetrica.symmetrica), 145

P

par_names() (sage.libs.singular.function.RingWrap method), 64

- Pari (class in *sage.rings.pari_ring*), 136
 pari_divisors_small() (in module *sage.libs.pari.convert_sage*), 134
 pari_is_prime() (in module *sage.libs.pari.convert_sage*), 134
 pari_is_prime_power() (in module *sage.libs.pari.convert_sage*), 134
 pari_maxprime() (in module *sage.libs.pari.convert_sage*), 135
 pari_prime_range() (in module *sage.libs.pari.convert_sage*), 135
 PariRing (class in *sage.rings.pari_ring*), 136
 parse_point_list() (in module *sage.libs.eclib.mwrank*), 28
 part_part_skewschur_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 145
 plethysm_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 145
 points() (*sage.libs.eclib.interface.mwrank_MordellWeil* method), 19
 poison_currRing() (in module *sage.libs.singular.ring*), 78
 pow_truncate() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 40
 print_currRing() (in module *sage.libs.singular.ring*), 78
 print_objects() (in module *sage.libs.ecl*), 10
 process() (*sage.libs.eclib.interface.mwrank_MordellWeil* method), 20
 pseudo_div() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 40
 pseudo_div_rem() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 41
 python() (*sage.libs.ecl.EclObject* method), 8
- Q**
- q_core_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 145
 qsieve() (in module *sage.libs.flint.qsieve_sage*), 43
- R**
- random_element() (*sage.rings.pari_ring.PariRing* method), 136
 random_partition_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 145
 rank() (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 13
 rank() (*sage.libs.eclib.interface.mwrank_MordellWeil* method), 22
 rank_bound() (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 14
 record_name_to_index() (*sage.libs.gap.element.GapElement_Record* method), 103
 regulator() (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 14
 regulator() (*sage.libs.eclib.interface.mwrank_MordellWeil* method), 23
 reset_default() (*sage.libs.singular.option.LibSingularOptions* method), 75
 reset_default() (*sage.libs.singular.option.LibSingularVerboseOptions* method), 77
 Resolution (class in *sage.libs.singular.function*), 62
 right_shift() (*sage.libs.flint.fmpz_poly_sage.Fmpz_poly* method), 41
 ring() (*sage.libs.singular.function.Converter* method), 62
 ring() (*sage.libs.singular.groebner_strategy.GroebnerStrategy* method), 79
 ring() (*sage.libs.singular.groebner_strategy.NCGroebnerStrategy* method), 80
 ring_cyclotomic() (*sage.libs.gap.element.GapElement_Ring* method), 104
 ring_finite_field() (*sage.libs.gap.element.GapElement_Ring* method), 104
 ring_integer() (*sage.libs.gap.element.GapElement_Ring* method), 105
 ring_integer_mod() (*sage.libs.gap.element.GapElement_Ring* method), 105
 ring_polynomial() (*sage.libs.gap.element.GapElement_Ring* method), 105
 ring_rational() (*sage.libs.gap.element.GapElement_Ring* method), 105
 ring_wrapper_Py (class in *sage.libs.singular.ring*), 78
 RingWrap (class in *sage.libs.singular.function*), 63
 rplacd() (*sage.libs.ecl.EclObject* method), 9
 rplacd() (*sage.libs.ecl.EclObject* method), 9
- S**
- sage() (*sage.libs.gap.element.GapElement* method), 94
 sage() (*sage.libs.gap.element.GapElement_Boolean* method), 95
 sage() (*sage.libs.gap.element.GapElement_Cyclotomic* method), 96
 sage() (*sage.libs.gap.element.GapElement_FiniteField* method), 97
 sage() (*sage.libs.gap.element.GapElement_Float* method), 97
 sage() (*sage.libs.gap.element.GapElement_Integer* method), 98
 sage() (*sage.libs.gap.element.GapElement_IntegerMod* method), 99
 sage() (*sage.libs.gap.element.GapElement_List* method), 101
 sage() (*sage.libs.gap.element.GapElement_Permutation* method), 102
 sage() (*sage.libs.gap.element.GapElement_Rational* method), 102

`sage()` (*sage.libs.gap.element.GapElement_Record method*), 104
`sage()` (*sage.libs.gap.element.GapElement_Ring method*), 105
`sage()` (*sage.libs.gap.element.GapElement_String method*), 106
`sage_matrix_over_ZZ()` (*sage.libs.eclib.mat.Matrix method*), 30
`sage_to_mpmath()` (*in module sage.libs.mpmath.utils*), 122
`sage.libs.ecl`
 module, 3
`sage.libs.eclib.constructor`
 module, 36
`sage.libs.eclib.homospace`
 module, 33
`sage.libs.eclib.interface`
 module, 11
`sage.libs.eclib.mat`
 module, 29
`sage.libs.eclib.mwrank`
 module, 27
`sage.libs.eclib.newforms`
 module, 31
`sage.libs.flint.arith_sage`
 module, 41
`sage.libs.flint.fmpq_poly_sage`
 module, 41
`sage.libs.flint.fmpz_poly_sage`
 module, 39
`sage.libs.flint.qsieve_sage`
 module, 43
`sage.libs.flint.ulong_extras_sage`
 module, 44
`sage.libs.gap.context_managers`
 module, 83
`sage.libs.gap.element`
 module, 92
`sage.libs.gap.gap_functions`
 module, 84
`sage.libs.gap.libgap`
 module, 85
`sage.libs.gap.saved_workspace`
 module, 107
`sage.libs.gap.test`
 module, 91
`sage.libs.gap.test_long`
 module, 84
`sage.libs.gap.util`
 module, 84
`sage.libs.giac`
 module, 45
`sage.libs.gsl.array`
 module, 51
`sage.libs.lcalc.lcalc_Lfunction`
 module, 53
`sage.libs.libecm`
 module, 49
`sage.libs.linbox.linbox_flint_interface`
 module, 109
`sage.libs.lrcalc.lrcalc`
 module, 111
`sage.libs.mpmath.utils`
 module, 119
`sage.libs.ntl.all`
 module, 125
`sage.libs.pari`
 module, 127
`sage.libs.pari.convert_sage`
 module, 130
`sage.libs.singular.function`
 module, 61
`sage.libs.singular.function_factory`
 module, 70
`sage.libs.singular.groebner_strategy`
 module, 79
`sage.libs.singular.option`
 module, 72
`sage.libs.singular.polynomial`
 module, 72
`sage.libs.singular.ring`
 module, 77
`sage.libs.singular.singular`
 module, 70
`sage.libs.symmetrica.symmetrica`
 module, 139
`sage.rings.pari_ring`
 module, 136
`saturate()` (*sage.libs.eclib.interface.mwrank_EllipticCurve method*), 14
`saturate()` (*sage.libs.eclib.interface.mwrank_MordellWeil method*), 23
`save()` (*sage.libs.singular.option.LibSingularOptions_abstract method*), 76
`scalarproduct_schubert_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 145
`scalarproduct_schur_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 146
`schur_schur_plet_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 146
`sdg_symmetrica()` (*in module sage.libs.symmetrica.symmetrica*), 146
`search()` (*sage.libs.eclib.interface.mwrank_MordellWeil method*), 26
`selmer_rank()` (*sage.libs.eclib.interface.mwrank_EllipticCurve method*), 15
`set_global()` (*sage.libs.gap.libgap.Gap method*), 89

- set_integer_from_gen() (in module *sage.libs.pari.convert_sage*), 135
 set_precision() (in module *sage.libs.eclib.mwrank*), 28
 set_printlevel() (in module *sage.libs.singular.function*), 66
 set_rational_from_gen() (in module *sage.libs.pari.convert_sage*), 136
 set_seed() (*sage.libs.gap.libgap.Gap* method), 90
 set_verbose() (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 16
 show() (*sage.libs.gap.libgap.Gap* method), 90
 shutdown_ecl() (in module *sage.libs.ecl*), 10
 si2sa_resolution() (in module *sage.libs.singular.singular*), 70
 si2sa_resolution_graded() (in module *sage.libs.singular.singular*), 71
 sign() (*sage.libs.eclib.homspace.ModularSymbols* method), 34
 silverman_bound() (*sage.libs.eclib.interface.mwrank_EllipticCurve* method), 17
 singular_function() (in module *sage.libs.singular.function*), 67
 SingularFunction (class in *sage.libs.singular.function*), 64
 SingularFunctionFactory (class in *sage.libs.singular.function_factory*), 70
 SingularKernelFunction (class in *sage.libs.singular.function*), 64
 SingularLibraryFunction (class in *sage.libs.singular.function*), 64
 skew() (in module *sage.libs.lrcalc.lrcalc*), 117
 sparse_hecke_matrix() (*sage.libs.eclib.homspace.ModularSymbols* method), 35
 specht_dg_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 start() (in module *sage.libs.symmetrica.symmetrica*), 146
 stirling_number_1() (in module *sage.libs.flint.arith_sage*), 43
 stirling_number_2() (in module *sage.libs.flint.arith_sage*), 43
 str() (*sage.libs.eclib.mat.Matrix* method), 30
 strict_to_odd_part_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 symbolp() (*sage.libs.ecl.EclObject* method), 9
- T**
- t_ELMSYM_HOMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_ELMSYM_MONOMIAL_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_ELMSYM_POWSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_ELMSYM_SCHUR_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_HOMSYM_ELMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_HOMSYM_MONOMIAL_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_HOMSYM_POWSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_HOMSYM_SCHUR_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_MONOMIAL_ELMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_MONOMIAL_HOMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_MONOMIAL_POWSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_MONOMIAL_SCHUR_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_POLYNOM_ELMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_POLYNOM_MONOMIAL_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_POLYNOM_POWER_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_POLYNOM_SCHUBERT_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 146
 t_POLYNOM_SCHUR_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_POWSYM_ELMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_POWSYM_HOMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_POWSYM_MONOMIAL_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_POWSYM_SCHUR_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_SCHUBERT_POLYNOM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_SCHUR_ELMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_SCHUR_HOMSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_SCHUR_MONOMIAL_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 t_SCHUR_POWSYM_symmetrica() (in module *sage.libs.symmetrica.symmetrica*), 147
 test_ecl_options() (in module *sage.libs.ecl*), 10
 test_integer() (in module *sage.libs.symmetrica.symmetrica*), 147
 test_loop_1() (in module *sage.libs.gap.test_long*), 84
 test_loop_2() (in module *sage.libs.gap.test_long*), 84
 test_loop_3() (in module *sage.libs.gap.test_long*), 84

`test_sigint_before_ecl_sig_on()` (in module `sage.libs.ecl`), 10
`test_write_to_file()` (in module `sage.libs.gap.test`), 91
`timestamp()` (in module `sage.libs.gap.saved_workspace`), 107
`truncate()` (`sage.libs.flint.fmpz_poly_sage.Fmpz_poly` method), 41
`two_descent()` (`sage.libs.eclib.interface.mwrank_EllipticCurve` method), 17

U

`unpickle_GroebnerStrategy0()` (in module `sage.libs.singular.groebner_strategy`), 80
`unpickle_NCGroebnerStrategy0()` (in module `sage.libs.singular.groebner_strategy`), 81
`unset_global()` (`sage.libs.gap.libgap.Gap` method), 91

V

`value()` (`sage.libs.lcalc.lcalc_Lfunction.Lfunction` method), 55
`var_names()` (`sage.libs.singular.function.RingWrap` method), 64
`vector()` (`sage.libs.gap.element.GapElement_List` method), 101

W

`workspace()` (in module `sage.libs.gap.saved_workspace`), 107

Z

`zero()` (`sage.libs.gap.libgap.Gap` method), 91
`zeta()` (`sage.rings.pari_ring.PariRing` method), 137